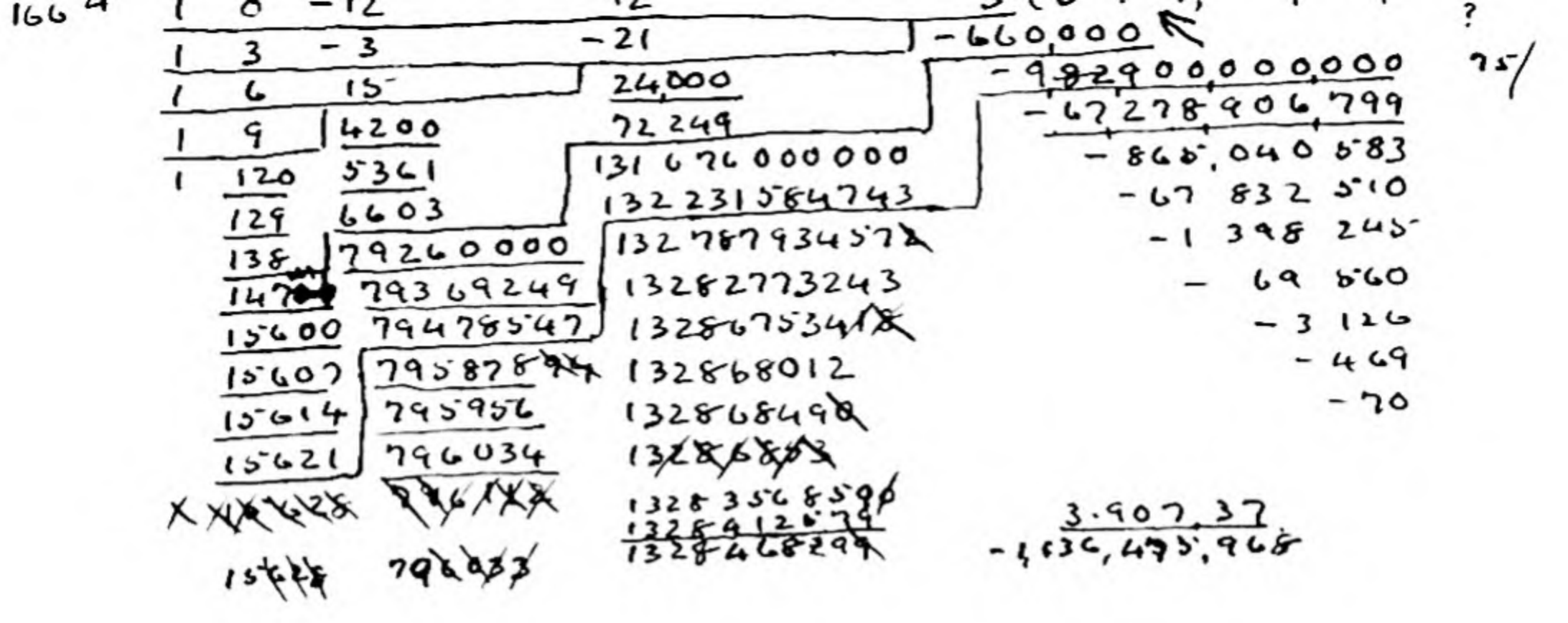

A TEXT BOOK OF
HIGHER ALGEBRA AND TRIGONOMETRY

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A TEXT BOOK OF

HIGHER ALGEBRA AND TRIGONOMETRY

by

B. B. BAGI

सत्वेषु मैत्री गुणिषु प्रमोदः। सङ्क्लश्यमानेषु कृपापरत्वम्॥
मध्यस्थभावो विपरीतवृत्तौ। सदा विधेयो विदुषा शिवाय ॥

अमितगति

One who is faithful to (mankind) and delights in worthy men, and regards generously the distressed (creatures) when faced by an unfavourable situation, he keeps his head. It is well for a scholar to act so.

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1927

TO

MY MOTHER

PREFACE

THE University of Bombay has recently introduced considerable changes in the mathematical course prescribed for the B. A., and B. Sc. examinations. The curriculum is now brought more into line with the rigorous methods of treatment adopted in modern books on Higher Mathematics. Thus the older works of Edwards and Williamson on the Calculus have given place to the newer and more rigorous treatises of Lamb and Gibson. There is also another change made namely in the grouping of the various subjects in the several papers. We now have Algebra and Trigonometry conveniently placed in one paper. There is, however, no easily available book which treats the course prescribed in these subjects concurrently, taking advantage of the common ground covered by them. Higher Algebra and Analytical Trigonometry touch each other and overlap in several places, and a great deal of unity of treatment can be secured and duplication of work avoided if these two subjects are dealt with together. The present work is an attempt to supply the need felt by students for such a combined and unified treatment.

A word of explanation is necessary regarding the plan of the book. Junior B. A. and B. Sc. students usually read Analytical Geometry in the first term. As a knowledge of determinants and algebraic equations is of great help in this study, these latter have been treated in the book earlier than is customary. On the other hand a satisfactory discussion of the theory of infinite series and of elementary functions presents great difficulty to the beginner and it is not convenient to take up the treatment of these subjects till the student has assimilated the ideas of limit, continuity and infinity and has had a thorough grounding in the principles of the Calculus. Accordingly they have been relegated to the concluding part of the book.

As regards fulness of treatment and variety of method the book should satisfy even the most ambitious student. With a view

to cure the student of the superstition that a mathematical theorem can be proved in one and only one way, many theorems have been proved by more than one method, even at the risk of increasing the bulk of the book: for example, as many as four methods are suggested for establishing the three typical series of chapter IX. The methods of the Calculus are suggested throughout. The geometrical aspect of complex numbers is brought more vividly to the notice of the student than is usually done. The author has in this way attempted to show to the student how the several branches of mathematics intermingle and dovetail into each other and how necessary it is for him to cultivate the habit of always keeping in view the whole field of mathematics. About two hundred examples, chosen mostly from university papers, are fully solved and about a thousand examples, a large number of which are original are set for solution. Some of the latter are hard and obviously they do not represent the university standard; hints for solution are given sometimes.

Almost every chapter ends with a list of books for reference most of them written by eminent authors. These references are mainly intended for the use of students who wish to read deeper into the subject or those who are willing to sacrifice part of their time for the joy of intellectual adventure. The author is grateful to the writers of the works mentioned in the book; the following books however deserve special me tion:

Bromwich, T. J. I. *Theory of Infinite Series*

Hardy, G. H. *A Course of Pure Mathematics*

Young, J. W *Fundamental Concepts of Algebra and Geometry*

Whitehead, A. N. *An Introduction to Mathematics.*

The author has received the greatest inspiration from the last book and recommends it strongly to the aspirants for mastery of mathematics. Dr. Whitehead deals with the most abstruse ideas that lie at the foundation of the science in a manner that is at once clear and non-technical, and the light style in which the book is written makes it easy and pleasant reading.

The author has great pleasure in acknowledging his deep indebtedness to Rev. C. P. SALDANHA, M. A., S. J., whose genius for teaching mathematics had won him a unique position in the student world. But for his sympathy, sustained encouragement, and personal interest in the author, the work would not have seen the light of day. His sincere thanks are due to his friend and former pupil, Wrangler D. C. PAWATE, B. A., (Bom. and Cantab.) who generously undertook the revision of last chapters and also to Mr. L. S. KAMAT, B. A. (Hons.) Daxina Fellow of the Karnatak College who went through the manuscript and verified the examples.

There are almost insuperable difficulties in printing works of this character in this country and the author's thanks are due to Mr. H. F. KATTIMANI, B. A., the manager of the TONTADARYA PRESS, DHARWAR for the gererous manner in which he has helped in the printing of this work.

Suggestions for improvement and notices of errors will be gratefully received by the author.

B. B. BAGI.

DHARWAR,
December 1927.

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CHAPTER 1

DISTRIBUTION OF PRODUCTS

1·1. To multiply $a+b$ by $l+m$, we multiply a by l and m and b by l and m, and the partial products so formed, are added algebraically. It is easily seen that the process is equivalent to this: Take a term with its proper sign from each of the factors $a+b$ and $l+m$ and multiply them. Do this in all possible ways without repetition. Add together all the partial products algebraically. Then the expression so formed is equal to the product $(a+b)\ (l+m)$. It is clear that the process can be extended to the distribution of the product of two factors each containing more than two terms. To obtain the product $(a+b)\ (l+m)\ (x+y)$, we multiply the first two factors and the result is multiplied by $x+y$. We see that the same result is obtained thus:—Take a term with its proper sign from each of the three factors and multiply them. Do this in all possible ways. Add together all the partial products algebraically. The expression thus formed, is the product $(a+b)\ (l+m)\ (x+y)$. It is thus evident that the process may be generalised so as to apply to products of any number of factors each containing any finite number of terms. This rule is known as the *Distributive Law*.

1·2. *Prove that*

$$(x+a_1)\ (x+a_2)\ (x+a_3)\ldots\ldots\ldots\ldots(x+a_n)$$
$$=x^n+\Sigma_1 x^{n-1}+\Sigma_2 x^{n-2}+\ldots\ldots\ldots\ldots\Sigma_{n-1} x+\Sigma_n$$

where Σr *denotes the sum of products of* $a_1, a_2, \ldots\ldots a_n$ *taken* r *at a time.*

Choosing x from each of the n factors, we have the term x^n. Since the choice can be made in one way only, the term x^n occurs only once in the distribution of the product. Choosing an a from a factor and x from each of the remaining $n-1$ factors, we have terms of the type $a_1 x^{n-1}$. The sum of all such products is clearly $\Sigma_1 x^{n-1}$. Choosing a's from two of the factors and x from

each of the remaining factors we get terms of the type $a_1 a_2 x^{n-2}$. Clearly we have a term of this type corresponding to every product of two factors chosen from a_1, a_2, ... a_n. Their sum is thus $\Sigma_2 x^{n-2}$. Proceeding in this way, we find that the distribution of the product contains groups of terms denoted by

$$x^n,\ \Sigma_1 x^{n-1},\ \Sigma_2 x^{n-2}, \ldots\ldots \Sigma_r x^{n-r}, \ldots\ldots \Sigma_{n-1} x, \Sigma_n .$$

Adding up all these terms, we have the identity

$$(x + a_1)(x + a_2)(x + a_3) \ldots\ldots\ldots\ldots (x + a_n)$$
$$= x^n + \Sigma_1 x^{n-1} + \Sigma_2 x^{n-2} + \Sigma_3 x^{n-3} + \ldots\ldots + \Sigma_{n-1} x + \Sigma_n .$$

Examble. Expand $(x-1)(x+2)(x+5)(x-6)$.

The coefficient of x^4 is 1.

The coefficient of x^3 is $-1+2+5-6 = 0$.

The coefficient of x^2 is

$$(-1)(2+5-6) + 2(5-6) - 5.6 = -33.$$

The coefficient of x is

$$(-1)\ 2.5 + (-1)\ 2\ (-6) + (-1)\ .5\ (-6) + 2.5(-6) = -28.$$

The coefficient of x^0 is $(-1)\ (2)\ (5)\ (-6) = 60$.

Hence the product is equal to

$$x^4 - 33\ x^2 - 28x + 60.$$

Examples. **I. a.**

Expand the following:

1. $(2x + 1)(3x - 2)(2x - 3)$;

2. $(x-1)(x+1)(2x-1)(2x+1)(3x-1)(3x+1)$;

3. $(x + a + b)(x - a + b)(x + a - b)(x - a - b)$.

1·3. *Prove the Binomial Theorem that*

$$(x + a)^n = x^n + {}_nC_1\ a x^{n-1} + {}_nC_2\ a^2 x^{n-2} + {}_nC_3\ a^3 x^{n-3} + \ldots\ldots + {}_nC_n\ a^n.$$

Put $a_1 = a_2 = a_3 = \ldots\ldots = a_n = a$ in the identity of §1·2. The left hand member reduces to $(x+a)^n$. Now each of the terms of Σ_r is the product of r a's; further, there are as many such products as there are combinations of n things taken r

at a time. Now by the theory of combinations, there are ${}_nC_r$ such products. Hence Σ_r reduces, after these substitutions are made, to ${}_nC_r\, a^r$. We thus establish the identity :—

$$(x+a)^n = {}_nC_0\, x^n + {}_nC_1\, a\, x^{n-1} + {}_nC_2\, a^2\, x^{n-2} + {}_nC_3\, a^3\, x^{n-3} + \ldots\ldots\ldots\ldots + {}_nC_n\, a^n.$$

Alternative method:—Since $x+a$ is linear and homogeneous in x and a, $(x+a)^n$ is homogeneous and of n dimensions in x and a. Its expansion therefore contains terms of the type $a^r x^{n-r}$. Now the term $a^r x^{n-r}$ may be regarded as formed by choosing a from r of the factors of

$$(x+a)\,(x+a)\,(x+a)\ldots\ldots\ldots\ldots(x+a)$$

and x from the remaining $n-r$ factors. This could be done in ${}_nC_r$ ways. Hence, the term $a^r\, x^{n-r}$ occurs ${}_nC_r$ times in the expansion of $(x+a)^n$. Now the different types of terms are

$$x^n,\quad a\,x^{n-1},\quad a^2\,x^{n-2},\ldots\ldots\, a^r\,x^{n-r},\ldots\ldots\, a^n.$$

Hence, we have the identity

$$(x+a)^n = x^n + {}_nC_1\, a\, x^{n-1} + {}_nC_2\, a^2\, x^{n-2} + \ldots\ldots\ldots + {}_nC_n\, a^n.$$

1·31. The student should note that in every term of the expansion of $(x+a)^n$, the suffix of C is equal to the exponent of a and that the sum of exponents of x and a is always n. Further, the exponent of a is 1 in the second term; 2 in the third term; 3 in the fourth term and so on. Thus the general term of the series or the $(r+1)$ th term is ${}_nC_r\, a^r\, x^{n-r}$ or

$$\frac{n\,!}{r\,!\,(n-r)\,!}\, a^r\, x^{n-r} = \frac{n(n-1)\,(n-2)\ldots(n-r+1)}{1.2.3.4\ldots\ldots r}\, a^r\, x^{n-r}.$$

We also see that $x+a$ and therefore $(x+a)^n$ is symmetrical with respect to x and a; hence the expansion of $(x+a)^n$ is also symmetrical with regard to x and a. Therefore the coefficients of $x^r\, a^{n-r}$ and $a^r\, x^{n-r}$ are equal. Thus $(x+a)^n$ is equal to

either $\quad x^n + {}_nC_1\, a\, x^{n-1} + \ldots\ldots\ldots + {}_nC_{n-1}\, a^{n-1}\, x + {}_nC_n\, a^n$

or $\quad {}_nC_n\, x^n + {}_nC_{n-1}\, a\, x^{n-1} + \ldots\ldots\ldots + {}_nC_1\, a^{n-1}\, x + a^n.$

Example 1. Expand $(2x - 3y)^5$.

We have the identity

$$(x + a)^5 = x^5 + {}_5C_1\, a\, x^4 + {}_5C_2\, a^2 x^3 + {}_5C_3\, a^3 x^2 + {}_5C_4\, a^4\, x + {}_5C_5\, a^5.$$

Putting $2x$ for x and $-3y$ for a, we have

$$(2x - 3y)^5$$

$$= (2x)^5 + {}_5C_1(-3y)(2x)^4 + {}_5C_2(-3y)^2(2x)^3 + {}_5C_3(-3y)^3(2x)^2 + {}_5C_4(-3y)^4(2x) + {}_5C_5(-3y)^5.$$

Substituting the values of ${}_5C_1$, ${}_5C_2$, ... and simplifying, we have

$$(2x - 3y)^5 = 32x^5 - 240x^4y + 720x^3y^2 - 1080x^2y^3 + 810xy^4 - 243y^5.$$

Example 2. Write down the 7th term in the expansion of $(2x - y)^{11}$.

The 7th term $= {}_{11}C_5\,(-y)^6(2x)^{11-6}$

$$= \frac{11\ 10.9.8.7.6}{6.5.4.3.2.1}\, 2^5 x^5 y^6 = 14784\, x^5 y^6.$$

Example 3. Find the term in x^5 in the expansion of $(x + x^{-1})^{11}$.

$(x + x^{-1})^{11} = x^{11}(1 + x^{-2})^{11}$. Thus the term in x^5 is obtained by multiplying x^{11} by the term in x^{-6} in the expansion of $(1 + x^{-2})^{11}$. Now by § 1·3

$$(1 + x^{-2})^{11} = 1 + {}_{11}C_1 x^{-2} + {}_{11}C_2 x^{-4} + {}_{11}C_3 x^{-6} + \ldots\ldots\ldots\ldots$$

Hence the required term is ${}_{11}C_3\, x^5$ or $165\, x^5$.

Example 4. Expand $(1 + x - 2x^2)^{10}$ in ascending powers of x as far as the term in x^3. [*B. U. 1925 Hons. I.*]

Applying the binomial theorem to $(1 + x - 2x^2)^{10}$ or to $\{1 + (x - 2x^2)\}^{10}$, we have

$$(1 + x - 2x^2)^{10} = 1 + {}_{10}C_1(x - 2x^2) + {}_{10}C_2(x - 2x^2)^2 + {}_{10}C_3(x - 2x^2)^3 + \ldots\ldots\ldots\ldots$$

Other terms of the expansion contain higher powers of x than the 3rd.

Now
$$1 = 1$$
$${}_{10}C_1(x - 2x^2) = 10x - 20x^2$$
$${}_{10}C_2(x - 2x^2)^2 = +45x^2 - 180x^3 + \ldots\ldots$$
$${}_{10}C_3(x - 2x^2)^3 = +120x^3 + \ldots\ldots$$

Hence $(1 + x - 2x^2)^{10} = 1 + 10x + 25x^2 - 60x^3 + \ldots\ldots$

Examples. I. b.

Expand the following :

1. $(x+1)^4 - (x-1)^4$, 2. $(x^2+1)^2 (x-1)^2$.

3. $(x+y+xy)^3$, 4. $(a-3b)^5$,

5. $(x-x^{-1})^6$, 6. $(x-3x^2)^5$.

7. Write down the 5th term in the expansion of $(x-2y)^{10}$.

8. Write down the middle term in the expansion of $(x-y)^{2n}$ (The expansion contains $2n+1$ terms.)

9. Write down the two middle terms of $(x-y)^{2n-1}$.

10. Which term of $(x^2-3x^{-1})^{21}$ is independent of x ?

11. Write down the term independent of x in $(2x-3x^{-1})^{2r}$.

Arrange according to powers of x. Expand the following as far as x^4 :

12. $(1-2x)^3 (1+x^2)^2$, 13. $(1-x)^3 (1+2x)^5$,

14. $(1+\sqrt{x+x^2})^7$, 15. $\left(1+\sqrt{\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}}\right)^5$.

16. Three consecutive coefficients in the expansion of $(1+x)^n$ are in the ratio 5 : 10 : 14. Show that $n = 11$.

17. If the coefficients of three consecutive terms in the expansion of $(1+x)^n$ are in A. P., prove that $n+2$ is a perfect square.

[r th, $(r+1)$ th and $(r+2)$ th coefficients are proportional to $1/(n-r+1)(n-r)$, $1/r(n-r)$, $1/(r+1)r$.
These are in A. P. if
$r(r+1) + (n-r+1)(n-r) = 2(r+1)(n-r+1)$.
This condition reduces to $n+2 = (n-2r)^2$].

Prove the identities in Examples 18 — 22:

18. $2^n \sin^{2n}\left[\frac{\pi}{4}+\theta\right] = {}_nC_0 + {}_nC_1 \sin 2\theta + {}_nC_2 \sin^2 2\theta + {}_nC_3 \sin^3 2\theta + \ldots\ldots + {}_nC_n \sin^n 2\theta$;

19. ${}_nC_1 - {}_nC_2(1+y) + {}_nC_3(1+y+y^2) - {}_nC_4(1+y+y^2+y^3) + \ldots$ to n terms $= (1-y)^{n-1}$;

20. $2^n a^n = (a+b)^n\left\{1 + {}_nC_1\left(\frac{a-b}{a+b}\right) + {}_nC_2\left(\frac{a-b}{a+b}\right)^2 + {}_nC_3\left(\frac{a-b}{a+b}\right)^3 + \ldots\ldots\right\}$;

21. $(2a)^n + (2b)^n = 2(a+b)^n\left\{1 + {}_nC_2\left(\frac{a-b}{a+b}\right)^2 + {}_nC_4\left(\frac{a-b}{a+b}\right)^4 + \ldots\right\}$;

22. $\mathrm{Cos}^n 2\theta$
$= (\cos^n\theta + {}_nC_2\cos^{n-2}\theta\sin^2\theta + {}_nC_4\cos^{n-4}\theta\sin^4\theta\ldots\ldots)^2$
$-({}_nC_1\cos^{n-1}\theta\sin\theta + {}_nC_3\cos^{n-3}\theta\sin^3\theta + \ldots\ldots)^2$;

23. If $(1+x)^n = C_0 + C_1 x + C_2 x^2 \ldots\ldots + C_n x^n$ prove that

$(C_0 + C_2 t + C_4 t^2 + C_6 t^3 \ldots)^2 - t(C_1 + C_3 t + C_5 t^2 + C_7 \ldots\ldots)^2$
$= C_0 - C_1 t + C_2 t^2 - C_3 t^3 + \ldots + (-1)^n C_n t^n$.

1·32. Investigate the numerically greatest term in the expansion of $(1+x)^n$.

Denote the numerical value of the rth term by t_r and the numerical value of x by X. Then

$$t_r = \frac{n(n-1)(n-2)\ldots\ldots(n-r+2)}{1.2.3.4\ \ldots\ldots\ \ (r-1)}\ X^{r-1} \qquad [\S 1{\cdot}31]$$

and $$t_{r+1} = \frac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{1.2.3.4\ldots\ldots\ldots\ldots\ldots\ldots(r-1)\,.\,r}\ X^r;$$

$$\therefore \qquad \frac{t_{r+1}}{t_r} = \frac{n-r+1}{r} X ,$$

$$\therefore \qquad t_{r+1} \begin{matrix} > \\ = \\ < \end{matrix} t_r \text{ according as } \frac{n-r+1}{r} X \begin{matrix} > \\ = \\ < \end{matrix} 1 ,$$

$$\therefore \qquad t_{r+1} \begin{matrix} > \\ = \\ < \end{matrix} t_r \text{ according as } r \begin{matrix} < \\ = \\ > \end{matrix} \frac{(n+1)X}{X+1} .$$

Case 1. If $(n+1)X/(1+X)$ is not an integer, let it be equal to $i+f$ where i is the greatest integer in it and f a positive fraction less than unity. Now for values 1, 2, 3,...i of r

$t_{r+1} > t_r$; $\therefore$ $t_2 > t_1$; $t_3 > t_2$:..... $t_{i+1} > t_i$.
Again for values $i+1$, $i+2$, $i+3$, ...of r

$t_{r+1} < t_r$ $\therefore$ $t_{i+1} > t_{i+2}$, $t_{i+2} > t_{i+3}$, ...
Thus in this case t_{i+1} is the numerically greatest term.

Case II. If $(n+1)X/(X+1)$ is an integer, denote it by i. Now for values 1, 2, 3, ... $i-1$ of r $t_{r+1} > t_r$; hence t_i is numerically greater than any term which precedes it. If $r = i$ $t_{i+1} = t_i$. Again for values $i+1$, $i+2$, ... of r $t_{r+1} < t_r$; hence all terms which follow t_{i+1} are numerically less than t_{i+1}. Thus in this case t_i and t_{i+1} are equal and are numerically greater than any other term.

Example. Find the greatest term or terms in the expansion of $(x+y)^{20}$ where $x = 3$ and $y = 4$.

Let t_r denote the r th term, then

$$\frac{t_{r+1}}{t_r} = \frac{20-r+1}{r} \cdot \frac{4}{3}$$

$$\therefore \qquad t_{r+1} \begin{matrix} > \\ = \\ < \end{matrix} t_r \text{ according as } 4(21-r) \begin{matrix} > \\ = \\ < \end{matrix} 3r$$

$$\therefore \qquad t_{r+1} \begin{matrix} > \\ = \\ < \end{matrix} t_r \text{ according as } r \begin{matrix} < \\ = \\ > \end{matrix} 12.$$

Thus $t_1 < t_2 < t_3 \ldots\ldots < t_{12}$, $t_{12} = t_{13}$ and $t_{13} > t_{14} > t_{15} \ldots\ldots$

$\therefore$ t_{12} and t_{13} are equal and are greater than any other term.

Examples. I. c..

Find the numerically greatest term or terms in the expansions of

1. $(x - y)^{25}$ where $x = 3$ and $y = 2$,

2. $(x - y)^{25}$ where $x = 2$ and $y = 3$,

3. $(x + y)^{40}$ where $x = 4$ and $y = 7$.

1·33. In the sequel, we give some properties of the binomial coefficients.

Prove that

(1) $ {}_nC_0 + {}_nC_1 + {}_nC_2 + {}_nC_3 + {}_nC_4 + \ldots\ldots\ldots + {}_nC_n = 2^n$

(2) $ {}_nC_0 - {}_nC_1 + {}_nC_2 - {}_nC_3 + {}_nC_4 - \ldots\ldots + (-1)^n {}_nC_n = 0$

and that

(3) $ {}_nC_0 + {}_nC_2 + {}_nC_4 + \ldots\ldots = {}_nC_1 + {}_nC_3 + {}_nC_5 + \ldots = 2^{n-1}$.

In the identity of §1·3, put $x = a = 1$. Then,

$$2^n = {}_nC_0 + {}_nC_1 + {}_nC_2 + {}_nC_3 + \ldots\ldots {}_nC_n . \qquad (1)$$

In the same identity putting $x = -a = 1$, we have

$$0 = {}_nC_0 - {}_nC_1 + {}_nC_2 - {}_nC_3 + {}_nC_4 - \ldots\ldots \qquad (2)$$

Adding (1) and (2), we have

$$2^n = 2 ({}_nC_0 + {}_nC_2 + {}_nC_4 + {}_nC_6 + \ldots\ldots\ldots)$$

or $$2^{n-1} = {}_nC_0 + {}_nC_2 + {}_nC_4 + {}_nC_6 + \ldots\ldots .$$

In the same way subtracting (2) from (1), we have

$$2^n = 2 ({}_nC_1 + {}_nC_3 + {}_nC_5 + \ldots\ldots)$$

or $$2^{n-1} = {}_nC_1 + {}_nC_3 + {}_nC_5 + \ldots\ldots$$

Example 1. Sum the series $\sigma_0 + {}_nC_1 \sigma_1 + {}_nC_2 \sigma_2 + {}_nC_3 \sigma_3 \ldots\ldots\ldots + {}_nC_n \sigma_n$ where $\sigma_r = 1 + r + r^2$.

We easily have $1 + r + r^2 = 1 + 2r + r(r-1)$

$\therefore$ ${}_nC_r \sigma_r = {}_nC_r + 2r\, {}_nC_r + r(r-1)\, {}_nC_r$

Now $r\,{}_nC_r = n\,\dfrac{(n-1)(n-2)\ldots(n-r+1)}{(r-1)(r-2)\ \ldots\ldots\ 2.1} = n.\ {}_{n-1}C_{r-1}.$

Similarly $r\,(r-1)\ {}_nC_r$

$= n(n-1)\ \dfrac{(n-2)(n-3)\ \ldots\ldots\ (n-r+1)}{(r-2)(r-3)\ \ldots\ldots\ 2.1} = n(n-1)\ {}_{n-2}C_{r-2}.$

Thus $\quad {}_nC_r\,\sigma_r = {}_nC_r + 2n.\ {}_{n-1}C_{r-1} + n(n-1)\ {}_{n-2}C_{r-2}.$

Putting $r = 0, 1, 2, 3, \ldots\ldots, n$ in this identity, we have

Successively $\quad \sigma_0 = {}_nC_0,$

$\sigma_1\ {}_nC_1 = {}_nC_1 + 2n.\ {}_{n-1}C_0,$

$\sigma_2\ {}_nC_2 = {}_nC_2 + 2n.\ {}_{n-1}C_1 + n(n-1).\ {}_{n-2}C_0,$

$\sigma_3\ {}_nC_3 = {}_nC_3 + 2n.\ {}_{n-1}C_2 + n(n-1)\ {}_{n-2}C_1,$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$\sigma_n\ {}_nC_n = {}_nC_n + 2n\ {}_{n-1}C_{n-1} + n(n-1)\ {}_{n-2}C_{n-2}.$

Adding up these relations, we have, by § 1·33, the sum of the given series

$= 2^n + 2n\,2^{n-1} + n(n-1)\,2^{n-2} = 2^{n-2}\,(n^2+3n+4).$

More easily Otherwise: Since $(1+x)^n = 1 + {}_nC_1x + {}_nC_2x^2 + \ldots + {}_nC_rx^r \ldots$

diff: a finite sequence

we have $\dfrac{d}{dx}(1+x)^n = n(1+x)^{n-1}$

$= {}_nC_1 + 2.\ {}_nC_2x + \ldots\ldots + r.\ {}_nC_r\,x^{r-1} + \ldots\ldots\ldots$

and $\dfrac{d^2}{dx^2}(1+x)^n = n(n-1)(1+x)^{n-2}$

$= 2.1.\ {}_nC_2 + \ldots\ldots\ldots + r(r-1)\ {}_nC_r\,x^{r-2} + \ldots\ldots\ldots;$

$\therefore \quad (1+x)^n + 2x\,n(1+x)^{n-1} + x^2\,n(n-1)(1+x)^{n-2}$

$= \ldots\ldots\ldots + \{\ 1 + 2r + r(r-1)\ \}\ {}_nC_r\,x^r + \ldots\ldots\ldots$

$= \sigma_0 + \sigma_1\ {}_nC_1\,x + \sigma_2\ {}_nC_2\,x^2 + \ldots\ldots\ldots + \sigma_r\ {}_nC_r\,x^r + \ldots\ldots\ldots.$

Putting $x = 1$ in this identity, we have the required result.

* Before we apply either of the two methods, it is necessary to change a polynomial like σ_r into the form

$$A_0 + A_1 r + A_2 r(r-1) + \ldots\ldots$$

As we shall have many occasions to use this transformation we give a method of effecting it. More generally, any polynomial $P(r)$ of the nth degree in r can be put into the form

$$b_0 + b_1(r-a_1) + b_2(r-a_1)(r-a_2) + b_3(r-a_1)(r-a_2)(r-a_3) + \ldots\ldots + b_n(r-a_1)(r-a_2)\ldots\ldots(r-a_n),$$

where $a_1, a_2, \ldots\ldots, a_n$ are arbitrary constants, any one of which except b_n, may be zero.

Suppose in the division of $P(r)$ by $r-a_1$, the quotient is $Q_1(r)$ and the remainder, b_0; we thus have

$$P(r) = b_0 + (r-a_1)Q_1(r). \qquad (1)$$

Suppose in the division of $Q_1(r)$ by $r-a_2$, the quotient is $Q_2(r)$ and the remainder, b_1; we thus have

$$Q_1(r) = b_1 + (r-a_2)Q_2(r). \qquad (2)$$

∴ from (1) and (2) we have

$$P(r) = b_0 + b_1(r-a_1) + (r-a_1)(r-a_2)Q_2(r). \qquad (3)$$

We thus see that, by repeating this operation, we can effect the required transformation. Much labour can be saved by applying the method of synthetic division (§ 5·1).

Example 2. If $(1+x)^n = a_0 + a_1 x + a_2 x^2 + \ldots\ldots + a_n x^n$, prove that

$$\frac{1}{2}a_0 2^2 + \frac{1}{3}a_1 2^3 + \frac{1}{4}a_2 2^4 + \ldots\ldots + \frac{1}{n+2}a_n 2^{n+2} = \frac{(2n+1)3^{n+1}+1}{(n+1)(n+2)}.$$

(*B. U. 1916 Hons. I*)

We have

$$\frac{1}{r+2}a_r 2^{r+2} = \frac{1}{r+2}\cdot\frac{n(n-1)\ldots(n-r+1)}{r(r-1)(r-2)\ldots 2.1}2^{r+2}$$

$$= \frac{r+1}{(n+1)(n+2)}\cdot\frac{(n+2)(n+1)n(n-1)\ldots(n-r+1)}{(r+2)(r+1)r(r-1)\ldots\ldots 2.1}2^{r+2}$$

$$= \frac{r+1}{(n+1)(n+2)}\ {}_{n+2}C_{r+2}2^{r+2}.$$

The given series thus reduces to

$$\frac{1}{(n+1)(n+2)}\left\{ {}_{n+2}C_2 2^2 + 2\,{}_{n+2}C_3 2^3 + \ldots \right.$$

$$\left. \ldots + (r+1)\,{}_{n+2}C_{r+2}2^{r+2} + \ldots \right\},$$

? a series of the same type as that of the last example; its sum can be therefore found by the method of that example.

Otherwise: It is easily seen that

$$\frac{1}{2}a_0 2^2 + \frac{1}{3}a_1 2^3 + \frac{1}{4}a_2 2^4 + \ldots + \frac{1}{n+2}a_n 2^{n+2}$$

$$= \int_0^2 a_0 x\,dx + \int_0^2 a_1 x^2\,dx + \int_0^2 a_2 x^3\,dx + \ldots + \int_0^2 a_n x^{n+1}\,dx$$

$$= \int_0^2 (a_0 x + a_1 x^2 + a_2 x^3 + \ldots + a_n x^{n+1})\,dx$$

$$= \int_0^2 x(1+x)^n\,dx$$

$$= \left[\frac{x}{n+1}(1+x)^{n+1}\right]_0^2 - \frac{1}{n+1}\int_0^2 (1+x)^{n+1}\,dx$$

[integrating by parts]

$$= \frac{2.3^{n+1}}{n+1} - \frac{1}{(n+1)(n+2)}(3^{n+2}-1)$$

$$= \frac{(2n+1)3^{n+1}+1}{(n+1)(n+2)}.$$

Example 3. In the notation of *Example* 2 prove that

$$a_0^2 + a_1^2 + a_2^2 + a_3^2 + a_4^2 + \ldots\ldots + a_n^2 = {}_{2n}C_n.$$

By § 1·31, we have

$$(1+x)^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots\ldots + a_r x^r + \ldots + a_n x^n$$

$$= a_n + a_{n-1}x + a_{n-2}x^2 + \ldots\ldots + a_r x^r + \ldots + a_0 x^n.$$

Now the coefficient of x^n in the product of the two right hand expansions is equal to

$$a_0^2 + a_1^2 + a_2^2 + a_3^2 \ldots\ldots\ldots + a_n^2 .$$

It is thus equal to the coefficient of x^n in $(1+x)^n \times (1+x)^n$ *i. e.* in $(1+x)^{2n}$. Hence it is equal to ${}_{2n}C_n$.

Example 4. If C_r is the coefficient of x^r in the expansion of $(1+x)^n$, prove that

$$2\frac{C_1}{1} - 2^2\frac{C_2}{2} + 2^3\frac{C_3}{3} - \ldots\ldots\ldots + (-1)^{n-1}2^n\frac{C_n}{n}$$

$$= 2 + \frac{2}{3} + \frac{2}{5} + \ldots\ldots\ldots + \frac{1}{n}\left\{1 + (-1)^{n-1}\right\}$$

and show how the value of

$$x\frac{C_1}{1} - x^2\frac{C_2}{2} + x^3\frac{C_3}{3} - x^4\frac{C_4}{4} + \ldots\ldots (-1)^{n-1}x^n\frac{C_n}{n}$$

may be found. [*B. U. 1914 Problems*]

It is easily seen that

$$2\frac{C_1}{1} - 2^2\frac{C_2}{2} + 2^3\frac{C_3}{3} - \ldots\ldots\ldots (-1)^{n-1}2^n\frac{C_n}{n}$$

$$= \int_0^2 C_1 dx - \int_0^2 C_2 x\,dx + \int_0^2 C_3 x^2 dx - \ldots\ldots\ldots$$

$$= \int_0^2 (C_1 - C_2 x + C_3 x^2 - \ldots\ldots\ldots)\,dx$$

$$= \int_0^2 \frac{1-(1-x)^n}{1-(1-x)}\,dx = \int_{-1}^{1}\frac{1-y^n}{1-y}\,dy \text{ where } 1-x=y$$

$$= \int_{-1}^{1}(1 + y + y^2 + \ldots\ldots y^{n-1})\,dy = \left[\frac{y}{1} + \frac{y^2}{2} + \frac{y^3}{3} + \ldots\frac{y^n}{n}\right]_{-1}^{1}$$

$$= \frac{2}{1} + \frac{2}{3} + \frac{2}{5} + \ldots\ldots + \frac{1}{n}\left\{1 + (-1)^{n-1}\right\}.$$

Similarly $x\frac{C_1}{1} - x^2\frac{C_2}{2} + x^3\frac{C_3}{3} - \ldots\ldots\ldots$

$$= \int_0^x \frac{1-(1-x)^n}{x}\,dx = -\int_1^{1-x}\frac{1-y^n}{1-y}\,dy \text{ where } y = 1-x.$$

$$= -\left[\frac{y}{1} + \frac{y^2}{2} + \frac{y^3}{3} + \ldots\ldots \frac{y^n}{n}\right]_1^{1-x}$$

$$= \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \left[\frac{(1-x)}{1} + \frac{(1-x)^2}{2} + \ldots + \frac{(1-x)^n}{n}\right].$$

Example 5. Show that the integral part of $(\sqrt{10}+3)^n$ is of the form $18r+1$ or $6r$ according as the integer n is even or odd. [*B. U. 1915 Problems.*]

Let $(\sqrt{10}+3)^n = I + F$ where I is the greatest integer contained in $(\sqrt{10}+3)^n$ and F a positive quantity less than unity.

Now $9<10<16 \quad \therefore\ 3<\sqrt{10}<4 \quad \therefore\ 0<\sqrt{10}-3<1$

$\therefore$ $(\sqrt{10}-3)^n$ is a positive quantity less than unity. Denote it by F_1.

Case I. If n is even, we have

$$(\sqrt{10}+3)^n + (\sqrt{10}-3)^n = 2\left(10^{\frac{n}{2}} + {}_nC_2\, 10^{\frac{n}{2}-1}\, 3^2 + \ldots\ldots\right).$$

$\therefore$ $I + F + F_1$ is an integer, $\therefore$ $F + F_1$ is an integer.

Now F and F_1 are positive fractions each less than unity; hence $F+F_1$ must be equal to 1.

$$\therefore\ I + 1 = 2\left(10^{\frac{n}{2}} + {}_nC_2\, 10^{\frac{n}{2}-1}\, 3^2 + {}_nC_4\, 10^{\frac{n}{2}-2}\, 3^4 + \ldots\ldots\right).$$

$$= 2\left(10^{\frac{n}{2}} - 1\right) + 2 + 2 \cdot 3^2\left({}_nC_2\, 10^{\frac{n}{2}-1} + {}_nC_4\, 10^{\frac{n}{2}-2}\, 3^2 + \ldots\ldots\ldots\right)$$

$$= 2\left(10^{\frac{n}{2}} - 1\right) + 2 + 18m \text{ where } m \text{ is an integer}$$

$$= 18r + 2; \qquad [\text{since } 10^{\frac{n}{2}} - 1 \text{ is divisible by } 10-1 \text{ or } 9].$$

$\therefore$ I is of the form $18r + 1$.

Case II. If n is odd, we have

$$(\sqrt{10}+3)^n - (\sqrt{10}-3)^n = 2.3\left(10^{\frac{n-1}{2}}\, {}_nC_1 + 10^{\frac{n-3}{2}}\, {}_nC_3 + \ldots\ldots\right).$$

Thus $I' + F' - F_1' = 6m$, where m is an integer.

Now F, F_1 being positive fractions each less than unity, their difference $F-F_1$ must be zero. Hence I or the integral part of $(\sqrt{10}+3)^n$ is of the form $6r$.

Examples 1. *d.*

1. If $(1+x)^{n-1} = a_0+a_1x+a_2x^2+\ldots+\ldots+a_{n-1}x^{n-1}$ and $(1+x)^n = b_0+b_1x+b_2x^2+\ldots+b_nx^n$ prove that $b_r = a_r+a_{r-1}$. Hence show how the coefficients in the expansions of $(1+x)^1$, $(1+x)^2$, $(1+x)^3$, $(1+x)^4$, ... can be successively written down.

If $(1+x)^n = c_0+c_1x+c_2x^2+c_3x^3+\ldots+\ldots+c_nx^n$ prove the identities in *Examples* 2—11.

2. $1c_1+2c_2x+3c_3x^2+4c_4x^3+\ldots+nc_nx^{n-1}=n(1+x)^{n-1}$.

3. $1.2c_2+2.3c_3x+\ldots+(n-1)nc_nx^{n-2}=(n-1)n(1+x)^{n-2}$.

4. $1^2c_1+2^2c_2x+3^2c_3x+\ldots+n^2c_nx^{n-1} = n(1+nx)(1+x)^{n-2}$.

5. $a^2c_0-(a+d)^2c_1+(a+2d)^2c_2-(a+3d)^2c_3+\ldots = 0$. $(n>2)$

6. $c_0^2-c_1^2+c_2^2-c_3^2+\ldots = 0$ or $(-1)^{\frac{n}{2}}c_{\frac{n}{2}}$ according as n is odd or [even.

7. (i) $c_0+2c_1x+3c_2x^2+\ldots+(r+1)c_rx^r+\ldots$
$=[1+(n+1)x](1+x)^{n-1}$.

(ii) $c_0+3c_2+5c_4+\ldots=2c_1+4c_3+6c_5+\ldots=(n+2).2^{n-2}$; $(n>1)$

(iii) $c_0^2+2c_1^2+3c_2^2+\ldots+(r+1)c_r^2+\ldots+(n+1)c_n^2$
$=_{2n}C_n+(n+1)_{2n-1}C_{n-1}$.

8. $1.c_1-(1+2)c_2+\ldots(-1)^{r-1}(1+2+3+\ldots+r)c_r+\ldots = 0$

where n is not less than three.

9. $\frac{c_0}{2}-\frac{c_1}{3}+\frac{c_2}{4}-\frac{c_3}{5}+\ldots\ldots=\frac{1}{(n+1)(n+2)}$.

10. $\frac{c_1}{1}-\frac{c_2}{2}+\frac{c_3}{3}-\frac{c_4}{4}+\ldots\ldots\ldots=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}$.

11. $\frac{c_1}{1^2}-\frac{c_2}{2^2}+\frac{c_3}{3^2}-\frac{c_4}{4^3}+\ldots$ to n terms $=\frac{1}{1}\frac{1}{1}+\left(\frac{1}{1}+\frac{1}{2}\right)\frac{1}{2}$
$+\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}\right)\frac{1}{3}+\ldots+\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots\frac{1}{n}\right)\frac{1}{n}$.

12. By considering the coefficient of x^r in the product of the expansions of $(1+x)^m$ and $(1+x)^n$, prove that

$$_{m+n}C_r = {}_mC_r\,.\,{}_nC_0 + {}_mC_{r-1}\,.\,{}_nC_1 + {}_mC_{r-2}\,.\,{}_nC_2 + {}_mC_{r-3}\,.\,{}_nC_3 + \ldots\ldots + {}_mC_0\,{}_nC_r.$$

1·4 The Multinomial Theorem. Smith 311. DM 135 for 3 vs

Prove that $(a + b + c + \ldots\ldots + l + m)^n$ *is equal to*

$$= \frac{n!}{p!\ q!\ r!\ \ldots\ldots t!\ u!}\ a^p.\ b^q.\ c^r.\ \ldots\ldots l^t\ m^u$$

where the series includes all terms corresponding to all positive integral solutions of the equation $p + q + r + \ldots + s + t + u = n$.

$a + b + c + \ldots\ldots + l + m$ is homogeneous and linear, therefore $(a + b + c + \ldots\ldots + l + m)^n$ is homogeneous and of n dimensions in $a, b, c, \ldots\ldots l, m$; hence its expansion contains terms of the type

$$a^p\ b^q\ c^r\ \ldots\ldots\ l^t\ m^u$$

where $p, q, r, \ldots\ldots$, are positive integers such that

$$p + q + r + \ldots + s + t + u = n.$$

Now expanding $(a + b + c + \ldots + l + m)^n$ by the binomial theorem, we have

$$[a + (b + c + \ldots + l + m)]^n = a^n + {}_nC_1\ a^{n-1}\ (b + c + \ldots) + {}_nC_2\ a^{n-2}\ (b + c + \ldots)^2 + \ldots.$$

We thus see that all terms of the required expansion which contain a^p as a factor are given by

$${}_nC_{n-p}\ a^p\ (b + c + d + \ldots\ldots + l + m)^{n-p}. \qquad (1)$$

Again applying the binomial theorem to $[b + (c + d + \ldots + l + m)]^{n-p}$ we have $(b + c + d + \ldots + l + m)^{n-p}$

$$= b^{n-p} + {}_{n-p}C_1\ b^{n-p-1}\ (c + d + \ldots\ldots l + m)^1 + \ldots$$

∴ all the terms of the expansion of $(b + c + d + \ldots)^{n-p}$ which contain b^q as a factor are given by

$${}_{n-p}C_{n-p-q}\ b^q\ (c + d + \ldots\ldots)^{n-p-q}. \qquad (2)$$

From (1) and (2) we deduce that all the terms of the expansion of $(a + b + \ldots\ldots)$ which contain $a^p\ b^q$ as a factor are given by

$${}_nC_{n-p}\ {}_{n-p}C_{n-p-q}\ a^p b^q\ (c + d + \ldots)^{n-p-q}.$$

Repeating this process, we ultimately find that the coefficient of the term $a^p\ b^q\ c^r\ \ldots\ldots\ l^t\ m^u$ is

$${}_nC_{n-p}\ {}_{n-p}C_{n-p-q}\ {}_{n-p-q}C_{n-p-q-r} \ldots {}_{n-p-q-\ldots-s}C_{n-p-q-r-\ldots-s-t}$$

or

$$\frac{n!}{p!\ q!\ r!\ \ldots\ldots l!\ u!}.$$

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Or, we may argue thus : p a's can be chosen from the n factors $(a+b+c \ldots\ldots +l+m)^n$ in ${}_nC_p$ ways ; q b's can be chosen from the remaining $n-p$ factors in ${}_{n-p}C_q$ ways and so on.

$\therefore$ the term $a^p\, b^q\, c^r \ldots\ldots l^t\, m^u$ occurs in the expansion

${}_nC_p \cdot {}_{n-p}C_q \cdot {}_{n-p-q}C_r \ldots\ldots$ or $\dfrac{n!}{p!\,q!\,r! \ldots\ldots t!\,u!}$ times.

Thus we get the required expansion.

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1·41 Show how to expand $(a+bx+cx^2+ \ldots +mx^k)^n$ into a polynomial in x.

By putting a, bx, $cx^2, \ldots mx^k$ for a, b, c, ... in the identity of § 1·4, we see that the general term of the required expansion is

$$\frac{n!}{p!\,q!\,r! \ldots\ldots t!}\, a^p\, (bx)^q\, (cx^2)^r \ldots\ldots (mx^k)^t$$

or $$\frac{n!}{p!\,q!\,r! \ldots t!}\, a^p\, b^q\, c^r \ldots\ldots x^{q\,+\,2r\,+\,3s\,+\,\ldots\ldots\,+\,kt}.$$

Cont'd Now to pick out terms in x^λ say, we have to subject $q, r, s \ldots t$ to the additional restriction that $q+2r+3s+ \ldots + kt = \lambda$.

Thus the coefficient of x^λ in the required expansion is given by

$$= \sum \frac{n!}{p!\,q!\,r! \ldots t!}\, a^p\, b^q\, c^r \ldots\ldots$$

where $p, q, r, \ldots t$ are positive integers (including zero) subject to the two conditions

$$p+q+r+ \ldots +t=n \quad \text{and} \quad q+2r+ \ldots +kt=\lambda.$$

Example 1. Find the coefficient of a^3b^2c in the expansion of $(a+b+c)^6$.

The required coefficient $= \dfrac{6!}{3!\,2!\,1!} = 60.$

Example 2. Find the coefficient of x^{11} in the expansion of $(1-3x^2+2x^3)^6$.

The general term in the expansion is given by

$$\frac{6!}{p!\,q!\,r!}\, 1^p\, (-3x^2)^q\, (2x^3)^r = \frac{6!}{p!\,q!\,r!}\, (-3)^q\, 2^r\, x^{2q\,+\,3r}$$

where p, q, r are subject to the two conditions

$$p+q+r=6 \quad (1) \qquad \textit{and} \qquad 2q+3r=11. \quad (2)$$

Since p, q, r, are positive integers, from (2), we see that only odd values of r are admissible, and that it cannot be greater than 3, for then q would be negative; hence r can assume only two values, 1 and 3. The corresponding values of p and q are 1, 4; 2, 1. The coefficient of x^{11} is therefore equal to

$$\frac{6!}{1!\,4!\,1!}\,2\,(-3)^4+\frac{6!}{2!\,1!\,3!}\,2^3\,(-3)^1=3420.$$

Verify this result by applying the method of *Example* 4. § 1·3.

Example 3. Expand $(a+b+c+\ldots+k+l)^3$.

Since the expansion must be homogeneous and of three dimensions in a, b, ..., the different types of terms that occur in it, are

$$a^3,\ a^2b,\ abc.$$

Now the coefficient of a^3 is $3!/3!0!\,0!$ or 1.

From symmetry, the coefficient of b^3, c^3, must also be equal to 1.

Again, the coefficient of a^2b is $3!/2!1! = 3$.

Similary the coefficients of terms like a^2c, b^2c, etc. are each equal to 3.

The coefficient of abc is $3!/1!1!1! = 6$.

Hence the required expansion is

$$\Sigma a^3+3\Sigma a^2b+6\Sigma abc.$$

Example 4. By using the identity $(1+2x+x^2)^n=(1+x)^{2n}$ prove that

$$\frac{n!\,2^t}{(n-t)!\,t!\,0!}+\frac{n!\,2^{t-2}}{(n-t+1)!\,(t-2)!\,1!}+\frac{n!\,2^{t-4}}{(n-t+2)!\,(t-4)!\,2!}+\ldots={}_{2n}C_t.$$

Now ${}_{2n}C_t$ is equal to the coefficient of x^t in the expansion of $(1+x)^{2n}$; so it must be equal to the coefficient of x^t in the expansion of $(1+2x+x^2)^n$. Now by § 1·41, the coefficient of x^t in the expansion of $(1+2x+x^2)^n$ is

$$\Sigma\frac{n!}{p!\,q!\,r!}\,2^q \quad (1)$$

where p, q, r are positive integers (including zero) subject to the two conditions

$$p+q+r=n \quad \textit{and} \quad q+2r=t.$$

Now setting $r=0, 1, 2, \ldots$, we easily have the sets of values

$$\left.\begin{matrix} p = n-t \\ q = t \\ r = o \end{matrix}\right\} \quad \left.\begin{matrix} n-t+1 \\ t-2 \\ 1 \end{matrix}\right\} \quad \left.\begin{matrix} n-t+2 \\ t-4 \\ 2 \end{matrix}\right\} \ldots\ldots$$

Substituting these values in (1), we have the required result.

Since q is positive, the greatest value of r is either $t/2$ or $(t-1)/2$ according as t is even or odd.

Example 5. If $(1+x+x^2)^n = a_0+a_1x+a_2x^2+\ldots+a_{2n}x^{2n}$, prove that (1) $a_0 = a_{2n}$, $a_1 = a_{2n-1}$, $a_2 = a_{2n-2}, \ldots$;

(2) $1 = a_0-a_1+a_2-\ldots-a_{2n-1}+a_{2n}$;

and that (3) $a_ra_0-a_{r-1}a_1+a_{r-2}a_2-a_{r-3}a_3+\ldots = 0$ or $a_{\frac{r}{2}}$

according as r is odd or even.

(1) Putting $1/x$ for x and then multiplying both sides of the given identity by x^{2n}, we have

$$(x^2+x+1)^n = a_0x^{2n}+a_1x^{2n-1}+a_2x^{2n-2}+\ldots+a_{2n}.$$

Now $(x^2+x+1)^n$ is the same as $(1+x+x^2)^n$. Thus we get

$$a_0+a_1x+a_2x^2+\ldots+a_{2n}x^{2n} = a_0x^{2n}+a_1x^{2n-1}+a_2x^{2n-2}+\ldots+a_{2n}$$

and so $a_0 = a_{2n}$, $a_1 = a_{2n-1}$, $a_2 = a_{2n-2}, \ldots$.

(2) Putting $x = -1$ in the given identity, we have

$$[1+(-1)+(-1)^2]^n = a_0+a_1(-1)+a_2(-1)^2+a_3(-1)^3+\ldots+a_{2n}(-1)^{2n}$$

or $$1 = a_0-a_1+a_2-a_3+a_4-\ldots.$$

(3) Changing x into $-x$ in the given identity, we have

$$(1-x+x^2)^n = a_0-a_1x+a_2x^2-a_3x^3+a_4x^4-\ldots.$$

Thus the coefficient of x^r in the product of the expansions of

$$(1+x+x^2)^n \text{ and } (1-x+x^2)^n$$

is $$a_1a_0-a_{r-1}a_1+a_{r-2}a_2-a_{r-3}a_3+\ldots+(-1)^ra_0a_r.$$

This quantity must therefore be equal to the coefficient of x^r in the expansion of $(1+x+x^2)^n \times (1-x+x^2)^n$ *i.e.* in that of $(1+x^2+x^4)^n$.

Now observing that $(1+x^2+x^4)^n$ can be derived from $(1+x+x^2)^n$ by changing x into x^2, we have

$$(1+x^2+x^4)^n = a_0+a_1x^2+a_2x^4+a_3x^6+\ldots+a_{2n}x^{4n}.$$

Hence the coefficient x^r in the expansion of $(1+x^2+x^4)^n$ is either zero or $a_{\frac{r}{2}}$ according as r is odd or even. Hence the required result.

Examples. 1 . e.

Find the coefficient of

1. x^6 in the expansion of $(1+2x+3x^2)^5$;
2. x^7 in the expansion of $(1-2x+2x^2)^{10}$;

3. $a^5 b c$ in the expansion of $(a-b+2c)^7$;

4. $a^4 b^3 c^3$ in the expansion of $(bc+ca+ab)^5$.

5. Prove that the coefficient of $a^p b^q c^r$ in the expansion of $(bc+ca+ab)^n$ is $n!/(n-p)!(n-q)!(n-r)!$ where p, q, r are positive integers such that $p+q+r = 2n$.

6. If x_1, x_2, x_3, x_4, denote $-a+b+c$, $a-b+c$, $a+b-c$, $a+b+c$ respectively, prove that

(I) $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 4(a^2+b^2+c^2)$,

(II) $x_1^3 + x_2^3 + x_3^3 - x_4^3 = -24\, abc$,

(III) $x_1^5 + x_2^5 + x_3^5 - x_4^5 = -80\, a b c\, (a^2+b^2+c^2)$,

(IV) $x_1^7 + x_2^7 + x_3^7 - x_4^7$

$= -56\, a b c\, [3(a^4+b^4+c^4) + 10\,(b^2c^2+c^2a^2+a^2b^2)]$.

In the following articles we establish identities which will be used, later on, to prove important trigonometrical results.

1·5. *To expand* $(a^n-b^n)/(a-b)$ *in terms of* $a+b$ *and* ab.

Denoting $a+b$ and ab by p and q, we have the obvious identity

$$\frac{1}{a-b}\left\{\frac{1}{1-ax}-\frac{1}{1-bx}\right\} = \frac{x}{1-(a+b)x+abx^2} = \frac{x}{1-px+qx^2}.$$

Now
$$\frac{1}{1-ax} = \frac{1-a^m x^m}{1-ax} + \frac{a^m x^m}{1-ax}$$

$$= 1+ax+a^2x^2+\ldots+a^n x^n+\ldots+a^{m-1}x^{m-1}+\frac{a^m x^m}{1-ax}.$$

Similarly
$$\frac{1}{1-bx} = 1+bx+b^2x^2+\ldots+b^n x^n+\ldots+b^{m-1}x^{m-1}+\frac{b^m x^m}{1-bx}$$

and also
$$\frac{1}{1-px+qx^2} = 1+(px-qx^2)+(px-qx^2)^2+\ldots$$

$$+(px-qx^2)^n+\ldots\ldots+(px-qx^2)^{m-1}+\frac{(px-qx^2)^m}{1-px+qx^2}.$$

Thus
$$\frac{1}{a-b}\left\{\frac{1}{1-ax}-\frac{1}{1-bx}\right\} = \frac{a-b}{a-b}x+\frac{a^2-b^2}{a-b}x^2+\frac{a^3-b^3}{a-b}x^3+\ldots$$

$$+\frac{a^{n-1}-b^{n-1}}{a-b}x^n+\ldots+\frac{1}{a-b}\left(\frac{a^m x^m}{1-ax}-\frac{b^m x^m}{1-bx}\right).$$

From this, we see that $(a^n-b^n)/(a-b)$ is equal to the coefficient of x^n in

$$x(px-qx^2)^{n-1}+x(px-qx^2)^{n-2}+x(px-qx^2)^{n-3}+\dots .$$

Hence $(a^n-b^n)/(a-b) = p^{n-1}-{}_{n-2}C_1\,p^{n-3}q+{}_{n-3}C_2\,p^{n-5}q^2-\dots$

$$\dots+(-1)^r{}_{n-r-1}C_r\,p^{n-2r-1}q^r\dots \quad . \quad (1)$$

It is sometimes convenient to rearrange the terms of the right hand expression of (1) in the reverse order.

If n is an even integer, equal to $2m$ say, we observe that the terms in the series (1) contain only odd powers of p. Now from (1) the term containing p^{2r-1} is the $\frac{n}{2}+r-1^{th}$ (for $2r-1 = 2m-2r'-1$, $\therefore r'-1 = 2m-m-r-1 = m+r-1$)

$$(-1)^{m-r}{}_{m+r-1}C_{m-r}\,p^{2r-1}\,q^{m-r} = (-1)^{m-r}{}_{m+r-1}C_{2r-1}p^{2r-1}q^{m-r} \quad (\S 1{\cdot}31)$$

$$= (-1)^{m-r}(m+r-1)(m+r-2)\dots(m+1)m(m-1)\dots(m-r+2)$$
$$\times(m-r+1)p^{2r-1}\,q^{m-r}/(2r-1)!$$

$$= (-1)^{m-r}m(m^2-1^2)(m^2-2^2)(m^2-3^2)\dots[m^2-(r-1)^2]p^{2r-1}q^{m-r}/(2r-1)!$$

Hence $(a^{2m}-b^{2m})/(a-b)$ — q in $m C_r \cdot m+r$ in such 2r+1

$$= (-1)^{m-1}\left\{ mpq^{m-1}-\frac{m(m^2-1^2)}{3!}p^3q^{m-2}+\frac{m(m^2-1^2)(m^2-2^2)}{5!}p^5q^{m-3}-\right.$$

$$\left.\dots+(-)^{r-1}\frac{m(m^2-1^2)(m^2-2^2)\dots[m^2-(r-1)^2]}{(2r-1)!}p^{2r-1}q^{m-r}+\dots\right\} \quad (2')$$

If n is an odd integer, equal to $2m+1$ say, we observe that the terms in the series (1) contain only even powers of p. Now from (1) the term containing p^{2r} is the $\frac{n+r}{2}+r-1^{th}$ (for $2r = 2m-2r'$, $\therefore n-r'-1 = 2m+1-m-r-1 = m+r$)

$$(-1)^{m-r}{}_{m+r}C_{m-r}p^{2r}q^{m-r} = (-1)^{m-r}{}_{m+r}C_{2r}p^{2r}q^{m-r}, \quad (\S 1{\cdot}31)$$

$$= (-1)^{m-r}(m+r)(m+r-1)\dots(m+1)m(m-1)\dots$$
$$\times(m-r+1)p^{2r}q^{m-r}/(2r)!$$

$$= (-1)^{m-r}(m+r)m(m^2-1^2)(m^2-2^2)\dots[m^2-(r-1)^2]p^{2r}q^{m-r}/(2r)!. \quad q/$$

Hence $(a^{2m+1}-b^{2m+1})/(a-b)$

$$= (-1)^m\left\{ q^m-\frac{(m+1)m}{2!}p^2q^{m-1}+\frac{(m+2)m(m^2-1^2)}{4!}p^4q^{m-2}-\dots\right.$$

$$\left.+(-)^r\frac{(m+r)m(m^2-1^2)\dots[m^2-(r-1)^2]}{(2r)!}p^{2r}q^{m-r}+\dots\right\} \quad . \quad (3')$$

Identities (2′) (3′) may be written in the form:

$(a^n-b^n)/(a-b)$ is equal to

$$(-1)^{\frac{1}{2}n-1}\left\{\frac{n}{1!2}pq^{\frac{1}{2}n-1}-\frac{n(n^2-2^2)}{3!2^3}p^3q^{\frac{1}{2}n-2}+\frac{n(n^2-2^2)(n^2-4^2)}{5!2^5}p^5q^{\frac{1}{2}n-3}\dots\right\} \quad (2)$$

when n is even; and

$$(-1)^{\frac{1}{2}n-1}\left\{q^{\frac{1}{2}n-1}-\frac{(n^2-1)}{2!2^2}p^2q^{\frac{1}{2}n-3}+\frac{(n^2-1)(n^2-3^2)}{4!2^4}p^4q^{\frac{1}{2}n-5}\dots\right\} \quad (3)$$

when n is odd.

1·6 In the same way, to express a^n+b^n in terms $a+b$ and ab, we use the obvious identity

$$\frac{1}{1-ax}+\frac{1}{1-bx}-1=\frac{1-abx^2}{1-(a+b)x+abx^2}=\frac{1-qx^2}{1-px+qx^2}.$$

Now proceeding as in § 1·5, we find that a^n+b^n is equal to the coefficient of x^n in

$$(1-qx^2)\,[1+(px-qx^2)+(px-qx^2)^2+\ldots+(px-qx^2)^{n-1}+(px-qx^2)^n]$$

Other terms are omitted as they do not involve terms in x^n.

Hence
$$a^n+b^n = p^n-{}_{n-1}C_1\,p^{n-2}q+{}_{n-2}C_2p^{n-4}q^2-{}_{n-3}C_3p^{n-6}q^3+\ldots$$
$$-q(p^{n-2}-{}_{n-3}C_1p^{n-4}q+{}_{n-4}C_2p^{n-6}q^2-\ldots)$$
$$=p^n-({}_{n-1}C_1+1)p^{n-2}q+({}_{n-2}C_2+{}_{n-3}C_1)p^{n-4}q^2-({}_{n-3}C_3+{}_{n-4}C_2)p^{n-6}q^3+$$
$$\ldots+(-1)^r({}_{n-r}C_r+{}_{n-r-1}C_{r-1})p^{n-2r}q^r+\ldots.$$
$$=p^n-\frac{n}{1!}p^{n-2}q+\frac{n(n-3)}{2!}p^{n-4}q^2-\frac{n(n-4)(n-5)}{3!}p^{n-6}q^3+$$
$$\ldots+(-1)^r\frac{n(n-r-1)(n-r-2)(n-r-3)\ldots(n-2r+1)}{r!}p^{n-2r}q^r+\ldots \quad (1)$$

As in §1·5 it is sometimes convenient to rearrange the terms of the series (1) in the reverse order *i.e.* in the ascending powers of p.

If n is an even integer, equal to $2m$ say, we observe that the right hand expression of (1) contains only even powers of p. Now the term which contains p^{2r} is from (1) equal to

$$(-1)^{m-r}[{}_{m+r}C_{m-r}+{}_{m+r-1}C_{m-r-1}]p^{2r}q^{m-r}$$
$$=(-1)^{m-r}[{}_{m+r}C_{2r}+{}_{m+r-1}C_{2r}]p^{2r}q^{m-r} \qquad (\S 1\cdot 31)$$
$$=(-1)^{m-r}2m(m+r-1)\ldots(m+1)m(m-1)\ldots(m-r+1)p^{2r}q^{m-r}/(2r)!$$
$$=(-)^{m-r}2m^2(m^2-1^2)(m^2-2^2)(m^2-3^2)\ldots[m^2-(r-1)^2]p^{2r}q^{m-r}/(2r)!.$$

Thus
$$a^{2m}+b^{2m}=(-1)^m2\left\{q^m-\frac{m^2}{2!}p^2q^{m-1}+\frac{m^2(m^2-1^2)}{4!}p^4q^{m-2}\right.$$
$$\left.\ldots+(-1)^r\frac{m^2(m^2-1^2)(m^2-2^2)\ldots[m^2-(r-1)^2]}{(2r)!}p^{2r}q^{m-r}+\ldots\right\}. \quad (2')$$

Similarly if n is an odd integer, equal to $2m+1$ say, we observe that the series (1) contains only odd powers of p. Now the term containing p^{2r+1} is, from (1), equal to

$$(-1)^{m-r}({}_{m+r+1}C_{m-r}+{}_{m+r}C_{m-r-1})p^{2r+1}q^{m-r}$$
$$=(-1^{m-r}({}_{m+r+1}C_{2r+1}+{}_{m+r}C_{2r+1})p^{2r+1}q^{m-r} \qquad (\S 1\cdot 31)$$
$$=(-1)^{m-r}(2m+1)(m+r)(m+r-1)\ldots(m+1)m(m-1)\ldots$$
$$\times(m-r+1)p^{2r+1}q^{m-r}/(2r+1)!$$

$$= (-1)^{m-r}(2m+1)(m+r)\,m(m^2-1^2)(m^2-2^2)\ldots \times [m^2-(r-1)^2]p^{2r+1}q^{m-r}/(2r+1)!$$

Hence $(-1)^m(a^{2m+1}+b^{2m+1})$ is equal to

$$(2m+1)\left\{ pq^m - \frac{m(m+1)}{3!}p^3q^{m-1} + \frac{m(m+2)(m^2-1^2)}{5!}p^5q^{m-2}\ldots \right.$$
$$\left. +(-1)^r\frac{m(m+r)(m^2-1^2)(m^2-2^2)\ldots[m^2-(r-1)^2]}{(2r+1)!}p^{2r+1}q^{m-r}+\ldots \right\} \quad (3')$$

Identities (2′) (3′) may also be written thus:

$$a^n+b^n = (-1)^{\frac{1}{2}n}2\left\{ q^{\frac{1}{2}n} - \frac{n^2}{2!2^2}p^2q^{\frac{1}{2}n-1} + \frac{n^2(n^2-2^2)}{4!2^4}p^4q^{\frac{1}{2}n-2}\ldots \right\} \quad (2)$$

when n is even; and

$$= (-1)^{\frac{1}{2}(n-1)}2\left\{ \frac{n}{1!2}pq^{\frac{1}{2}(n-1)} - \frac{n(n^2-1^2)}{3!2^3}p^3q^{\frac{1}{2}(n-3)} + \frac{n(n^2-1^2)(n^2-3^2)}{5!2^5}p^5q^{\frac{1}{2}(n-5)}-\ldots \right\} \quad (3)$$

when n is odd.

1·61 The following identity is immediately obtained by putting $a=z$ and $b=z^{-1}$ in (1) of § 1·6. We thus have

$$z^n+\bar{z}^n = (z+z^{-1})^n - \frac{n}{1}(z+z^{-1})^{n-2} + \frac{n(n-3)}{1.2}(z+z^{-1})^{n-4}-\ldots.$$

which is a polynomial in $z+z^{-1}$. We may derive corresponding formulae from the results of § 1·5.

Examples. I. Prove that:

1. $(a+b)^5-a^5-b^5 = 5ab(a+b)(a^2+ab+b^2)$;
2. $(a+b)^7-a^7-b^7 = 7ab(a+b)(a^2+ab+b^2)^2$;
3. $(a+b)^{11}-a^{11}-b^{11} = 11ab(a+b)(a^2+ab+b^2) \times [(a+b)^6-3ab(a+b)^4+4a^2b^2(a+b)^2-a^3b^3]$,
4. $(a+b)^{13}-a^{13}-b^{13} = 13ab(a+b)(a^2+ab+b^2)^2 \times [(a+b)^6-3(a+b)^4ab+5(a+b)^2a^2b^2-a^3b^3]$.

5. By expansions of $(1-2x-3x^2)^{-1}$ or otherwise, prove that

$$2^n+\frac{(n-1)}{1!}3.2^{n-2}+\frac{(n-2)(n-3)}{2!}3^2.2^{n-4}+\frac{(n-3)(n-4)(n-5)}{3!}3^3.2^{n-6}+\ldots = \tfrac{1}{4}[3^{n+1}+(-1)^n].$$ [*B. U. 1925 Hons. I.*]

[$(1-2x-3x^2)^{-1} = 1/(1-3x)(1-(-x))$; hence the required result is obtained by putting $a=3, b=-1$, and $n+1$ for n in (1) of § 1·5.]

6. Prove that

$$\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\} \div \sqrt{5}$$

$$= 1 + \frac{(n-2)}{1} + \frac{(n-3)(n-4)}{1.2} + \frac{(n-4)(n-5)(n-6)}{1.2.3} + \ldots .$$

[The right hand series can be derived from that of (1) § 1·5. by putting $p = +1$ and $q = -1$]

Examples. I. g.

If $(1+x)^n = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n$ prove that

1. $n c_n^2 + (n-1) c_{n-1}^2 + (n-2) c_{n-2}^2 + \ldots = n \cdot {}_{2n-1}C_{n-1}$;

2. $\frac{c_0 x}{1} + \frac{c_1 x^2}{2} + \frac{c_2 x^3}{3} + \frac{c_3 x^4}{4} + \ldots + \frac{c_n x^{n+1}}{n+1} = \frac{(1+x)^{n+1} - 1}{n+1}$.

3. $\frac{c_0 x}{1^2} + \frac{c_1 x^2}{2^2} + \frac{c_2 x^3}{3^2} + \frac{c_3 x^4}{4^2} + \ldots + \frac{c_n x^{n+1}}{(n+1)^2}$

$$= \frac{1}{n+1}\left\{\frac{(x+1)-1}{1} + \frac{(x+1)^2-1}{2} + \frac{(x+1)^3-1}{3} + \ldots + \frac{(x+1)^{n+1}-1}{n+1}\right\}.$$

4. Prove that $(3+\sqrt{5})^n + (3-\sqrt{5})^n$ is equal to

$$6^n - \frac{n}{1} 6^{n-2} 4 + \frac{n(n-3)}{1.2} 6^{n-4} 4^2 - \frac{n(n-4)(n-5)}{1.2.3} 6^{n-6} 4^3 + \ldots .$$

Hence or otherwise shew that the integer next greater than $(3+\sqrt{5})^n$ is divisible by 2^n.

5. Prove that the integral part of $(5+\sqrt{21})^n$ is of the form $2^n M - 1$ where M is an integer.

6. Prove that the integral part of $(\sqrt{13}+3)^{2n+1}$ is of the form 3.2^{2n+1} M where M is an integer.

7 If $(\sqrt{13}+3)^{2n+1} = I + F$ where I is the greatest integer contained in $(\sqrt{13}+3)^{2n+1}$, prove that $F(I+F) = 4^{2n+1}$.

8 If $(1+x+x^2+x^3)^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_{3n} x^{3n}$, prove that,

(1.) $a_0 = a_{3n}$, $a_1 = a_{3n-1}$, $a_2 = a_{3n-2}$, ... ;

(2.) $a_0 = 1$, $a_1 = n/1!$, $a_2 = n(n+1)/2!$, $a_3 = n(n+1)(n+2)/3!$,

$a_r = {}_nC_r + {}_nC_{r-2} \times {}_nC_1 + {}_nC_{r-4} \times {}_nC_2 + {}_nC_{r-6} \times {}_nC_3 + \ldots$.

[Use the identity $(1+x)^n (1+x^2)^n = (1+x+x^2+x^3)^n$]

(3) $a_0 + a_2 + a_4 + a_6 + \ldots = a_1 + a_3 + a_5 + \ldots = 2^{2n-1}$.

9 If the product of any number of factors of the form

$(1+x+x^2+\ldots x^{2p})^l, (1+x+x^2+\ldots x^{2q})^m \ldots$ where $p, l, q, m, \ldots$, are positive integers, is expanded into an expression of the form

$$a_0+a_1x+a_2x^2+a_3x^3+\ldots+a_rx^r+\ldots$$

prove that

(1) the coefficients equidistant from the two ends are equal;

(2) $a_1a_0-a_{r-1}a_1+a_{r-2}a_2-\ldots=0$ or $a_{\frac{r}{2}}$ according as r is odd or even;

(3) $a_0-a_1+a_2-a_3+\ldots+(-1)^ra_r+\ldots=1$.

10. Show that 9 can be expressed as the sum of integers 1, 2, 3, 4, 5, 6, 7, 8 in 7 ways; an integer occurs not more than once as a term in the sum. [Consider the coefficient of x^9 in the distribution of the product $(1+x)(1+x^2)(1+x^3)(1+x^4)\ldots(1+x^8)$.]

11. Prove that weights weighing 1, 3, 9, 27 lbs. can be manipulated to deal out any quantity weighing up to 40 lbs. in integral lbs. [Distribute $(x+x^{-1}+1)(x^3+x^{-3}+1)(x^9+x^{-9}+1)(x^{27}+x^{-27}+1)$]

12. Prove that 7 coins worth 1, 2, 2^2, 2^3, 2^4, 2^5, 2^6, 2^7 rupees can be manipulated to pay in integral rupees, any amount up to Rs. 255.

13. Prove that, if $1+y=(1+x)^n$ and y^r where r is a positive integer, be expanded in ascending powers of x, the coefficient of x^t in the expansion is

$$\sum_{s=0}^{s=r-1}(-1)^s{}_rC_s\times{}_{\overline{r-s}n}C_t\,.$$

[*B. U. 1924 Pass. I.*]

14. Prove that

$${}_nC_{n-t}-{}_nC_1\times{}_{n-1}C_{n-t}+{}_nC_2\times{}_{n-2}C_{n-t}-\ldots=0.$$

15. Prove that

$$\frac{n!}{(n-t)!t!}\left(1-\frac{t}{n-1}\cdot\frac{n-t}{1}+\frac{t}{n-2}\cdot\frac{n-t-1}{1}\cdot\frac{n-t-2}{1}-\ldots\ldots\right)=\frac{n!}{t!}(-1)^t$$

and $[x=-1]$ $\quad a_0-a_1+a_2-a_3+\ldots\ldots+a_{2n}=1$

where

$$a_r=\frac{n!}{0!r!(n-r)!}+\frac{n!}{1!(r-2)!(n-r+1)!}+\frac{n!}{2!(r-4)!(n-r+2)!}+\ldots.$$

CHAPTER II

DETERMINANTS

2·1 This chapter is devoted to the discussion of an important class of functions that occur so often in many branches of mathematics. It will be found that they secure much elegance of form and great economy of labour.

2·2 Definition of Determinants. Consider the four quantities arranged in the form of a square

$$a_1 \quad b_1$$
$$a_2 \quad b_2 \quad .$$

Quantities a_1, b_1, a_2, b_2 are called *elements* or *constituents*. Elements in a horizontal line are said to form a *row*; they are characterized by the same suffix. Those in a vertical line form a *column*; they are characterized by the same alphabet. (Thus a_1, b_1; a_2, b_2 are two rows and a_1, a_2; b_1, b_2, are two columns.) We operate upon these four elements in the following manner :—Form all possible products of elements choosing one and only one element from each column, and one and only one element from each row. We have thus a_1b_2, a_2b_1; arrange the factors in alphabetical order and then assign to them positive or negative sign according as the number of inversions in the order of suffixes is even or odd. [A greater suffix preceding a smaller one gives rise to an *inversion*]. We thus have $+a_1b_2$ and $-a_2b_1$. Adding up algebraically all the terms so formed, we finally obtain the expression.

$$a_1b_2 - a_2b_1.$$

We denote this symbolically by

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \quad \text{or} \quad [a_1\, b_2]\,,$$

It is called a determinant of the second *order* as it contains only two rows and only two columns.

Consider similarly the nine elements arranged in the form of a square

$$\begin{matrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{matrix}$$

We have three rows consisting of elements a_1, b_1, c_1; a_2, b_2, c_2; and a_3, b_3, c_3, and three columns consisting of a_1, a_2, a_3; b_1, b_2, b_3; and c_1, c_2, c_3. The products formed by taking one and only one element from each row and one and only one element from each column, are the following:– $a_1b_2c_3$, $a_1b_3c_2$, $a_2b_1c_3$, $a_2b_3c_1$, $a_3b_1c_2$, $a_3b_2c_1$. The factors of each product are arranged in alphabetical order. In $a_1b_2c_3$ there is no inversion in the order of suffixes, hence its sign is positive. In $a_1b_3c_2$, 3 precedes 2; hence there is one inversion and so we assign to it a negative sign. In $a_2b_3c_1$, 2 and 3 precede 1, therefore there are two inversions and so its sign is positive. $a_3b_2c_1$ contains three inversions; for 3 precedes 2 and 1; and 2 precedes 1; hence its sign is negative. The signs of $a_2b_1c_3$ and $a_3b_1c_2$ may be similarly found to be – and + respectively. These products with their proper signs, when added, give rise to the expression

$$a_1b_2c_3-a_1b_3c_2+a_2b_3c_1-a_2b_1c_3+a_3b_1c_2-a_3b_2c_1.$$

It is called a determinant of the third order and is symbolically denoted by

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \text{or} \quad [\, a_1\ b_2\ c_3 \,] \ .$$

2·21 As there is a one-to-one correspondence between rows and suffixes and columns and ~~alphabets~~, the above rule may be modified thus:– Form products by choosing only one element from each column taken in order from left to right subject to the condition that one and only one element is to be chosen from each row. Positive or negative sign is given to the product according as there are an even or an odd number of inversions in the order of rows from which elements are successively chosen.

2·22 Yet another modification is the following:– Arrange all letters in alphabetical order and then permute suffixes in all possible ways. Corresponding to every permutation we have a term of the determinant, its sign being determined by the number of inversions in the order of suffixes as before. Thus we have

seen that in a determinant of the second order, there are 2! or 2 terms, and that in a determinant of the third order there are 3! or 6 terms. In the same way we can define a determinant of the nth order. It contains n elements in each of its rows and n elements in each of its columns. Its expansion contains $n!$ terms, some of which may reduce to zero due to zero elements.

The notation of arranging the elements in the form of a square and then enclosing them between two vertical lines is due to Cauchy (1789–1857) It furnishes a very good example of how a happy notation facilitates the development of a mathematical subject.

The following are illustrations of problems involving determinants.

Example 1. The condition that the equations $a_1x+b_1y=0$ and $a_2x+b_2y=0$ should be consistent is

$$a_1b_2-a_2b_1 \equiv \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0.$$

Example 2. It is easily seen that the ratios $x : y : z$ determined by the equations $a_1x+b_1y+c_1z=0$ and $a_2x+b_2y+c_2z=0$ are given by

$$\frac{x}{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}} = \frac{y}{\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}.$$

Example 3. It is well known that the area of a $\triangle$ XYZ whose vertices are given by (a_1, b_1) (a_2, b_2) and (a_3, b_3) is, (axes being rectangular)

$$\tfrac{1}{2}\left[a_1(b_2-b_3)+a_2(b_3-b_1)+a_3(b_1-b_2) \right]$$

or

$$\tfrac{1}{2}\begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix}.$$

Example 4. Show that $+a_4b_3c_2d_1$ is a term of the determinant $[a_1b_2c_3d_4]$.

Each of the letters occurs once only. Hence only one element is taken from each column. Similarly it involves each of the suffixes only once. Hence one and only one element is taken from each row. Further all the letters are arranged in alphabetical order. We also see that the number of inversions is 6. Therefore $+a_4b_3c_2d_1$ is a term of the determinant with its proper sign.

Examples. II. a.

1. Prove that $a_2b_1c_4d_3$, $a_3b_2c_4d_1$, $-a_3b_2c_1d_4$, $-a_2b_4c_1d_3$, $-d_1b_2c_3a_4$ are the terms of $[a_1\ b_2\ c_3\ d_4]$.

2. Are the products $a_1b_2c_5d_4c_3$, $a_2b_1c_4d_3e_5$ and $a_1b_2c_2d_5e_5$ terms of the determinant $[a_1\ b_2\ c_3\ d_4\ e_5]$?

3. Write down, with their proper signs, all the terms in the determinant $[a_1\ b_2\ c_3\ d_4]$ which contain as a factor (1) $a_1b_2d_3$ (ii) $a_2b_1c_4$ (iii) a_4d_1 and (iv) a_1d_4.

4. Prove that the terms of $[a_1\ b_2]$ multiplied by those of $[c_3\ d_4]$ give some of the terms of $[a_1\ b_2\ c_3\ d_4]$.

5. Find all the terms, with their proper signs, of the determinant $[a_1\ b_2\ c_3\ d_4\ e_5]$, which contain $a_1d_2b_3$ as a factor.

6. Prove that $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$.

Evaluate the following determinants:

7. $\begin{vmatrix} ka_1 & lb_1 \\ ka_2 & lb_2 \end{vmatrix}$;

8. $\begin{vmatrix} ka_1 + lb_1 & mb_1 \\ ka_2 + lb_2 & mb_2 \end{vmatrix}$;

9. $\begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix}$;

10. $\begin{vmatrix} 6 & 4 \\ 7 & 4 \end{vmatrix}$;

11. $\begin{vmatrix} 1 & 8 & 3 \\ 4 & 5 & 6 \\ 7 & 2 & 9 \end{vmatrix}$;

12. $\begin{vmatrix} 1 & 5 & 3 \\ 4 & -1 & 6 \\ 7 & -7 & 9 \end{vmatrix}$;

13. $\begin{vmatrix} a_1 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ 0 & 0 & c_3 \end{vmatrix}$;

14. $\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ 0 & b_2 & 0 & 0 \\ 0 & b_3 & c_3 & d_3 \\ 0 & b_4 & 0 & d_4 \end{vmatrix}$.

15. Prove that $(-1)^{\frac{1}{2}n(n-1)}\ a_n\ b_{n-1}\ c_{n-2}\ d_{n-3} \ldots k_1$ is a term of the determinant $[a_1\ b_2\ c_3\ d_4 \ldots k_n]$.

16. Prove that the condition that the equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

should represent two straight lines may be written in the from

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0.$$

2·3 *A determinant is not changed when columns are written as rows or rows as columns.*

We shall prove, for instance, that the two determinants

$$\Delta_1 \equiv \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \text{and} \quad \Delta_2 \equiv \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

are identical. The argument will be found to be perfectly general.

Consider a term $+a_2b_3c_1$ of Δ_1. Writing it so that all its suffixes are arranged in the ascending or normal order of magnitude, we have $+c_1a_2b_3$.(i) Now the term so derived is numerically equal to a term of Δ_2; for, since all suffixes occur each once only and are arranged in ascending order of magnitude, the product is formed by taking one and only one element from each column of Δ_2 taken in order. Also each letter occurs once only; hence it is formed by taking one and only one element from each row of Δ_2.(ii) Further the number of inversions in the alphabetical order of letters in the derived term $c_1a_2b_3$ is the same as the number of inversions in the order of suffixes in $a_2b_3c_1$, each inversion in the order of suffixes in $a_2b_3c_1$ giving rise to an inversion in the alphabetical order of letters in $c_1a_2b_3$. Therefore $+c_1a_2b_3$ is a term of Δ_2 with its proper sign (§2·21). In the same way, every other term of Δ_1 is a term of Δ_2 with its proper sign. It can be proved that reciprocally every term of Δ_2 is a term of Δ_1 with its proper sign. Hence we have

$$\Delta_1 \equiv \Delta_2 .$$

The argument is perfectly general, and can be applied to determinants of any order. It follows as a consequence of this theorem that any theorem which holds for rows, holds also for columns and vice versa.

2·4 *If two suffixes in a term of a determinant are interchanged, the alphabetical order of letters remaining the same, we get a new term of the determinant and the two terms have opposite signs.*

Let the determinant be $[a_1\, b_2\, c_3 \ldots l_n]$. Suppose the given term is $Pd_h\, Qg_k\, R$ where P, Q, R, stand for products of elements. By interchanging h and k we get Pd_kQg_hR. Since in Pd_hQg_kR. every letter and every suffix occurs once only, it is clear, that

the same is true of Pd_kQg_hR. Therefore Pd_kQg_hR is numerically equal to a term of the determinant.(iv) It remains to prove that the signs of the terms are different. For this, it is sufficient to show that the difference between the numbers of inversions in the two terms is an odd integer. Suppose that p of the suffixes in Q are greater than both h and k; q, less than both h and k; and r, between h and k. Suppose further that $h > k$. Now it is clear that the number of inversions due to any suffix in P or R is the same in both terms. (i) h is greater than k and also greater than $q+r$ suffixes in Q; corresponding to each of these $q+r+1$ suffixes there is one more inversion in the given term than in the derived term.(ii) Again r of the suffixes in Q are greater than k and less than h. Due to each of these r suffixes, there is one more inversion in the first term than in the second.(iii) Further in the derived term since k is greater than q of the suffixes in Q there are q more inversions in the derived term than in the original. All other inversions are not disturbed by the interchange of h and k. Hence the original term contains $q+r+1+r-q$ or $2r+1$ inversions more than the derived one. The two terms therefore must have opposite signs. If h were less than k, $2r+1$ inversions would be more in the derived term than in the original. Thus in either case the two terms differ in sign. Hence the theorem.

Example. Determine the sign of $a_4b_3c_2d_6e_5f_1$ of the determinant $[a_1\ b_2\ c_3\ d_4\ e_5\ f_6]$.

If we interchange 1 and 4; 2 and 3; and 4 and 6 normal order of suffixes is restored. Hence the sign of the term is negative.

Examples. II. b.

Determine the signs of terms:

(1) $a_5b_4c_3d_2e_1$ and $a_3b_2c_1e_4d_5$ of $[a_1\ b_2\ c_3\ d_4\ e_5]$;

(2) $f_5b_6c_4d_3e_2a_1$ and $a_6b_5c_3d_4e_1f_2$ of $[a_1b_2c_3d_4e_5f_6]$.

2·41 *If two rows or columns of a determinant are interchanged, then the resulting determinant differs in sign only from the given determinant.*

By changing the hth and kth rows of the determinant

$$\Delta_1 \equiv \begin{vmatrix} a_1 & b_1 & c_1 & \dots & l_1 \\ a_2 & b_2 & c_2 & \dots & l_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_h & b_h & c_h & \dots & l_h \\ \dots & \dots & \dots & \dots & \dots \\ a_k & b_k & c_k & \dots & l_k \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix}$$

we have another determinant

$$\Delta_2 \equiv \begin{vmatrix} a_1 & b_1 & c_1 & \dots & l_1 \\ a_2 & b_2 & c_2 & \dots & l_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_k & b_k & c_k & \dots & l_k \\ \dots & \dots & \dots & \dots & \dots \\ a_h & b_h & c_h & \dots & l_h \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix}.$$

Consider any term Pa_hQg_kR of Δ_1 where P, Q, R, stand, as in § 2·4, for products of elements. Choosing elements from corresponding places in Δ_2, we have clearly the term Pa_kQg_hR of Δ_2. This term must involve the same number of inversion in the order of rows in Δ_2 as the term Pa_hQg_kR does in Δ_1. Hence the two terms must agree in sign. Farther by § 2·4, Pa_hQg_kR is a term of Δ_2 obtained by interchanging the suffixes h and k in Pa_kQg_hR of Δ_2; hence Pa_hQg_kR and Pa_kQg_hR must differ in sign. Thus, corresponding to every term of Δ_1 we have enother term of Δ_2 differing from it only in sign. It is clear that this relation is reciprocal. Hence the given determinant and the transformed determinant must be identical except in sign.

It is evident that a corresponding theorem holds with respect to columns. The two foregoing theorems expedite the determination of the sign of a term of a determinant.

2·411 *If two or more rows (or columns) of a determinant are identical then its value is zero.*

This is an immediate deduction from § 2·41.

Denote the given determinant by Δ and the determinant obtained by interchanging any two of the identical rows (or columns) by Δ_1. Now by § 2·41 $\Delta = -\Delta_1$ but clearly Δ_1 is identically the same as Δ. Hence $\Delta = -\Delta$; $\therefore \quad \Delta = 0$.

Example 1. Prove that

$$-\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = \begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix} = -(b-c)(c-a)(a-b).$$

Denote the determinant as usual by Δ. It is clearly a polynomial in a. If b is put for a, two of its rows become identical and therefore it vanishes for this substitution. Hence $a-b$ is a factor of Δ. Similarly $a-c$ is also a factor. Considering the determinant as a polynomial in c, we similarly find that $b-c$ is also a factor. Now Δ is of three dimensions. Hence

$$\Delta = k\,(b-c)(c-a)(a-b).$$

where k is a number to be suitably chosen. The coefficients of a^2b on both sides of the equation are $+1$ and $-k$, hence $k=-1$; and so

$$\Delta = -(b-c)(c-a)(a-b)$$

Examples. II. c.

Evaluate the determinants:

1. $\begin{vmatrix} a^2+a & a & 1 \\ b^2+b & b & 1 \\ c^2+c & c & 1 \end{vmatrix}$;

2. $\begin{vmatrix} (a+x)^2 & a+x & 1 \\ (b+x)^2 & b+x & 1 \\ (c+x)^2 & c+x & 1 \end{vmatrix}$;

3. $\begin{vmatrix} m_2m_3 & m_1 & 1 \\ m_3m_1 & m_2 & 1 \\ m_1m_2 & m_3 & 1 \end{vmatrix}$;

4. $\begin{vmatrix} m_1^3 & m_2^3 & m_3^3 \\ m_1 & m_2 & m_3 \\ 1 & 1 & 1 \end{vmatrix}$;

5 $\begin{vmatrix} 1 & y+z & yz \\ 1 & z+x & zx \\ 1 & x+y & xy \end{vmatrix}$;

6 $\begin{vmatrix} 1 & y+z & xy+xz \\ 1 & z+x & yz+yx \\ 1 & x+y & xz+zy \end{vmatrix}$

7. Prove that

$$\begin{vmatrix} a_1 & b_1 & c_1 & \dots \\ a_2 & b_2 & c_2 & \dots \\ \dots & \dots & \dots & \dots \\ a_n & b_n & c_n & \dots \end{vmatrix} = (-1)^{\frac{1}{2}n(n-1)} \begin{vmatrix} a_n & b_n & c_n & \dots \\ a_{n-1} & b_{n-1} & c_{n-1} & \dots \\ \dots & \dots & \dots & \dots \\ a_2 & b_2 & c_2 & \dots \\ a_1 & b_1 & c_1 & \dots \end{vmatrix}.$$

2·5 Development of a Determinant. If from a determinant of the nth order p rows and p columns are suppressed the remaining elements form a determinant of the $n-p$th order. It is called *a* pth minor. It is usual to denote a determinant by Δ; a first

minor is then denoted by Δ_x where x is the element occuring at the intersection of the suppressed row and column. Thus in the determinant $[\, a_1\ b_2\ c_3 \,]$

$$\Delta_{a_1} = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} \text{ and } \Delta_{c_2} = \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$$

are the first minors corresponding to the elements a_1 and c_2.

2·6 *In the usual notation, prove that the determinant.*

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$$

is equal to

(I) $$a_1\Delta_{a_1} - a_2\Delta_{a_2} + a_3\Delta_{a_3} - a_4\Delta_{a_4}\,.$$

and (II) $$(-1)^3\,[d_1\,\Delta_{d_1} - d_2\,\Delta_{d_2} + d_3\,\Delta_{d_3} - d_4\,\Delta_{d_4}]\,.$$

Since one and only one element is selected from each row and one and only one from each column, to form a term of a determinant, it follows that it must be a homogeneous and linear function of the elements of a row or of a column. Thus the given determinant must be an expression of the form

$$a_1A_1 + a_2A_2 + a_3A_3 + a_4A_4 \qquad (1)$$

where A_1, A_2, A_3, A_4 are independent of a_1, a_2, a_3, a_4. Now, if P is a term, with its proper sign, of the first minor Δ_{a_1}, then a_1P is a term of the given determinant; for in a_1P one and only element from each row and one and only one from each column is present, and the number of inversions in a_1P is the same as in P. Conversely, if a_1P' is a term of the determinant, then P' contains one and only one element from each row and from each column other than the row and the column passing through a_1 *i. e.* one and only one element from each row and column of the first minor Δ_{a_1}. Further the number of inversions in a_1P' is the same as that in P'. Hence P' must be a term of Δ_{a_1} with its proper sign; thus we have

$$A_1 = \Delta_{a_1}\,.$$

In the same way it is easily seen that a term Q of Δ_{a_2} multiplied by a_2 is equal in numerical value to and differs in sign from

a term of the determinant— In a_2Q, 2 precedes 1 and hence there is one more inversion in a_2Q than in Q. On the other hand, if a_2Q' is a term of the determinant, then Q' must be equal in numerical value to a term of $\triangle_{a_2}$ but the number of inversions in a_2Q' is greater than that of inversions in Q' by 1 and hence a_2Q' and Q' must differ in sign. Thus we have

$A_2 = (-1)^1 \triangle_{a_2}$. Similarly, $A_3 = (-1)^2 \triangle_{a_3}$ and $A_4 = (-1)^3 \triangle_{a_4}$. Making these substitutions for A_1, A_2, A_3, A_4 in (1), we get the required result (I).

Let us now try to expand the determinant in terms of the elements of any other column, say the 4th. By interchanging the elements of the 4th column with those of the 3rd and then with those of the 2nd and so on, the given determinant transforms into

$$(-1)^3 \begin{vmatrix} d_1 & a_1 & b_1 & c_1 \\ d_2 & a_2 & b_2 & c_2 \\ d_3 & a_3 & b_3 & c_3 \\ d_4 & a_4 & b_4 & c_4 \end{vmatrix} = (-1)^3 \triangle' \text{ say.}$$

Now applying the first part and expanding $\triangle'$ in terms of the elements of its 1st column, we have

$$\triangle' = d_1 \triangle'_{d_1} - d_2 \triangle'_{d_2} + d_3 \triangle'_{d_3} - d_4 \triangle'_{d_4} . \qquad (2)$$

But it is easily seen that $\triangle'_{d_1}$, $\triangle'_{d_2}$, etc. are identically the same as $\triangle_{d_1}$, $\triangle_{d_2}$ etc. Making these substitutions in (2), we immediately have the required result (II).

Similarly the determinant can be expanded in terms of the elements of any row or any column. The argument is quite general and may be extended to determinants of any order.

A_1, B_2,...etc. are called the *prepared minors* or *cofactors* corresponding to the elements a_1, b_2,...respectively. We see, from above, that they are numerically equal to $\triangle_{a_1}$, $\triangle_{b_2}$ ... .

The following corollaries are important; they simplify very much the manipulation and reduction of determinants.

2·61. Cor. 1. *The sum of the elements of a column (or a row) multiplied by the cofactors of corresponding elements of any other column (or row) is zero.*

For instance, we shall prove that in the case of the determinant of § 2·6,

$$d_1A_1+d_2A_2+d_3A_3+d_4A_4=0\,.$$

By §2·6, we have the identity

$$\Delta \equiv a_1A_1+a_2A_2+a_3A_3+a_4A_4 \qquad (1)$$

where A_1, A_2, A_3, A_4 are independent of a_1, a_2, a_3, a_4. Setting d_1, d_2 d_3, d_4 in this identity, we have

$$\begin{vmatrix} d_1 & b_1 & c_1 & d_1 \\ d_2 & b_2 & c_2 & d_2 \\ d_3 & b_3 & c_3 & d_3 \\ d_4 & b_4 & c_4 & d_4 \end{vmatrix} = d_1A_1+d_2A_2+d_3A_3+d_4A_4. \qquad (2)$$

Now the left-hand determinant of (2) has two identical columns and hence, by §2·411, it is equal to zero. The required result is now obvious.

2·62. Cor. 2. *If the elements of a column (or a row) are multiplied by the same quantity x the resulting determinant is x times the given determinant.*

For instance by § 2·6, we have the identity

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1A_1+a_2A_2+a_3A_3\,.$$

Substituting a_1x, a_2x, a_3x for a_1, a_2, a_3 in this identity and observing that A_1, A_2, A_3 are independent of a_1, a_2, a_3, we have

$$\begin{vmatrix} xa_1 & b_1 & c_1 \\ xa_2 & b_2 & c_2 \\ xa_3 & b_3 & c_3 \end{vmatrix}$$

$$=xa_1A_1+xa_2A_2+xa_3A_3=x(a_1A_1+a_2A_2+a_3A_3)=x.\Delta.$$

This result may be easily generalised ;

thus
$$\begin{vmatrix} la_1 & mb_1 & nc_1 \\ la_2 & mb_2 & nc_2 \\ la_3 & mb_3 & nc_3 \end{vmatrix} = l\begin{vmatrix} a_1 & mb_1 & nc_1 \\ a_2 & mb_2 & nc_2 \\ a_3 & mb_3 & nc_3 \end{vmatrix}$$

$$=lm\begin{vmatrix} a_1 & b_1 & nc_1 \\ a_2 & b_2 & nc_2 \\ a_3 & b_3 & nc_3 \end{vmatrix} = lmn\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

Example. Prove that

$$\Delta = \begin{vmatrix} a & 1 & bc \\ b & 1 & ca \\ c & 1 & ab \end{vmatrix} = \begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix}.$$

Multiplying the rows of Δ by a, b, c, respectively, we have

$$abc\,\Delta = \begin{vmatrix} a^2 & a & abc \\ b^2 & b & abc \\ c^2 & c & abc \end{vmatrix}.$$

Now by § 2·62 removing abc from the last column, we have

$$\begin{vmatrix} a^2 & a & abc \\ b^2 & b & abc \\ c^2 & c & abc \end{vmatrix} = abc \begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix}$$

and thus we have the required result.

2·63. Cor. 3. *If to the elements of a column (or a row) the corresponding elements of any number of other columns (or rows) or the same multiples of them be added, the resulting determinant is equal to the original one.*

For instance if to the elements of the first column of

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \tag{1}$$

we add l and m times the elements of the 2nd and the 3rd columns, we have then

$$\begin{vmatrix} a_1+l\,b_1+mc_1 & b_1 & c_1 \\ a_2+l\,b_2+mc_2 & b_2 & c_2 \\ a_3+l\,b_3+mc_3 & b_3 & c_3 \end{vmatrix}. \tag{2}$$

Now by § 2·6 we have the identity

$$[\,a_1\,b_2\,c_3\,] \equiv a_1A_1+a_2A_2+a_3A_3.$$

Setting $a_1+lb_1+mc_1$, $a_2+lb_2+mc_2$ and $a_3+lb_3+mc_3$ for a_1, a_2, a_3, respectively in this identity and observing that A_1, A_2, A_3 are independent of a_1, a_2, a_3, we find that the determinant (2) is equal to

$$(a_1+lb_1+mc_1)A_1+(a_2+lb_2+mc_2)A_2+(a_3+lb_3+mc_3)A_3$$

or to

$$(a_1A_1+a_2A_2+a_3A_3)+l(b_1A_1+b_2A_2+b_3A_3)+m(c_1A_1+c_2A_2+c_3A_3).$$

But by § 2·6 and § 2·61 we have

$$b_1A_1+b_2A_2+b_3A_3 = c_1A_1+c_2A_2+c_3A_3 = 0;$$

and thus the two determinants (1) and (2) are identical.

2·64. Cor. 4. *If each element of a column (or a row) consists of the sum of the same number (k say) of terms then the determinant can be expressed as the sum of k determinants.*

For example we shall prove that the determinant

$$\begin{vmatrix} a_1+a_1'+a_1'' & b_1 & c_1 \\ a_2+a_2'+a_2'' & b_2 & c_2 \\ a_3+a_3'+a_3'' & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1' & b_1 & c_1 \\ a_2' & b_2 & c_2 \\ a_3' & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1'' & b_1 & c_1 \\ a_2'' & b_2 & c_2 \\ a_3'' & b_3 & c_3 \end{vmatrix}.$$

Putting $a_1+a_1'+a_1''$, $a_2+a_2'+a_2''$, $a_3+a_3'+a_3''$ for a_1, a_2, a_3 in the identity

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1A_1+a_2A_2+a_3A_3 \qquad (1)$$

we find that the given determinant is equal to

$$(a_1+a_1'+a_1'')A_1+(a_2+a_2'+a_2'')A_2+(a_3+a_3'+a_3'')A_3$$

or to $$(a_1A_1+a_2A_2+a_3A_3)+(a_1'A_1+a_2A_2+a_3A_3)$$
$$+(a_1''A_1+a_2''A_2+a_3''A_3).$$

Putting a_1', a_2', a_3', and a_1'', a_2'', a_3'' for a_1, a_2, a_3 respectively in the same identity (1) we have

$$[a_1' \quad b_2 \quad c_3] = a_1'A_1+a_2'A_2+a_3'A_3$$

and $$[a_1'' \quad b_2 \quad c_3] = a_1''A_1+a_2''A_2+a_3''A_3.$$

Thus making these substitutions, we have finally the desired result.

It is easy to generalise the result thus :

If elements of certain columns consist of sums of p, q, r ..., terms then the determinant can be expressed as the sum of pqr determinants.

For instance the determinant

$$\begin{vmatrix} a_1+a_1' & b_1+b_1' & c_1 \\ a_2+a_2' & b_2+b_2' & c_2 \\ a_3+a_3' & b_3+b_3' & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1+b_1' & c_1 \\ a_2 & b_2+b_2' & c_2 \\ a_3 & b_3+b_3' & c_3 \end{vmatrix} + \begin{vmatrix} a_1' & b_1+b_1' & c_1 \\ a_2' & b_2+b_2' & c_2 \\ a_3' & b_3+b_3' & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1' & c_1 \\ a_2 & b_2' & c_2 \\ a_3 & b_3' & c_3 \end{vmatrix} + \begin{vmatrix} a_1' & b_1 & c_1 \\ a_2' & b_2 & c_2 \\ a_3' & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1' & b_1' & c_1 \\ a_2' & b_2' & c_2 \\ a_3' & b_3' & c_3 \end{vmatrix}.$$

The first three of the following examples illustrate three important applications of determinants. The student is advised to study them carefully.

Eaample 1. Solve the linear simultaneous equations

$a_1x+b_1y+c_1z=d_1$, $a_2x+b_2y+c_2z=d_2$ and $a_3x+b_3y+c_3z=d_3$.

Multiply the equations by A_1, A_2, A_3 where A_1, A_2, A_3 represent cofactors of a_1, a_2, a_3 in the determinant $[a_1\ b_2\ c_3]$ and then add them. We then have

$$x(a_1A_1+a_2A_2+a_3A_3)+y(b_1A_1+b_2A_2+b_3A_3)$$
$$+ z(c_1A_1+c_2A_2+c_3A_3) = d_1A_1+d_2A_2+d_3A_3.$$

Now by §2·6 and §2·61,

$a_1A_1+a_2A_2+a_3A_3 = [a_1\ b_2\ c_3]$, $b_1A_1+b_2A_2+b_3A_3 = 0$,

$c_1A_1+c_2A_2+c_3A_3 = 0$ and $d_1A_1+d_2A_2+d_3A_3 = [d_1\ b_2\ c_3]$

and hence $x\,[a_1\ b_2\ c_3] = [d_1\ b_2\ c_3]$ or $x = [d_1\ b_2\ c_3] \div [a_1\ b_2\ c_3]$ if $[a_1\ b_2\ c_3] \neq 0$.

Similarly $y = [a_1\ d_2\ c_3] \div [a_1\ b_2\ c_3]$ and $z = [a_1\ b_2\ d_3] \div [a_1\ b_2\ c_3]$

The above method can be applied to solve any set of linear simultaneous equations. cf. 5·8 2·1

Example. 2. Find the condition that the following equations should be consistent:

$$a_rx+b_ry+c_rz = d_r \quad (r = 1, 2, 3, 4).$$

Assuming that $[a_1\ b_2\ c_3] \neq 0$ we have from the first three equations (see *Example* 1 above)

$x = [d_1\ b_2\ c_3] \div [a_1\ b_2\ c_3]$; $y = [a_1\ d_2\ c_3] \div [a_1\ b_2\ c_3]$; *etc.*

Substituting these values in the last equation, we get the required condition

$$a_4\,[d_1\ b_2\ c_3] + b_4\,[a_1\ d_2\ c_3] + c_4\,[a_1\ b_2\ d_3] = d_4\,[a_1\ b_2\ c_3].$$

This may be put into the form

$$a_4A_4+b_4B_4+c_4C_4+d_4D_4 = 0 \text{ or } [a_1\ b_2\ c_3\ d_4] = 0.$$

Otherwise:– Multiply the four equations by A_1, A_2, A_3, A_4 where A_1, A_2, A_3, A_4 (supposed to be different from zero) are cofactors of a_1, a_2, a_3, a_4 in the determinant $[a_1,\ b_2\ c_3\ d_4]$ and then add.

By using results of §§ 2·6, 2·6, we get $x\,[a_1\,b_2\,c_3\,d_4] = 0$; and thus since x, y, z, cannot all be zero, we have $[a_1\,b_2\,c_3\,d_4] = 0$.

Example 6. Eliminate x between the two quadratic equations

$$ax^2+bx+c=0 \text{ and } a_1x^2+b_1x+c_1=0$$

We have
$$ax^3+bx^2+cx+0 = 0$$
$$0 \quad +ax^2+bx+c = 0$$
$$a_1x^3+b_1x^2+c_1x+0 = 0$$
and
$$0+a_1x^2+b_1x+c_1 = 0\,.$$

Treating x^3, x^2, x as independent variables and applying the method of the last example, we have

$$\begin{vmatrix} a & b & c & o \\ o & a & b & c \\ a_1 & b_1 & c_1 & o \\ o & a_1 & b_1 & c_1 \end{vmatrix} = 0\,.$$

The method employed here is known as *Sylvester's Dialytic Method of Elimination*. To eliminate x between, the two equations $f(x)=0$, $\psi(x)=0$ of degrees m and n say, we multiply the first equation by $x^{m-1}, x^{m-2}, x^{m-3}, \ldots x$, 1 and the second, by $x^{n-1}, x^{n-2}, x^{n-3} \ldots x$, 1 and thus obtain $m+n$ equations. From these $m+n$ equations we can eliminate $m+n-1$ quantities $x^{m+n-1}, x^{m+n-2}, x^{m+n-3} \ldots x^2, x$.

Example 4. Evaluate the determinant formed by the magic square

$$\begin{vmatrix} 11 & 24 & 7 & 20 & 3 \\ 4 & 12 & 25 & 8 & 16 \\ 17 & 5 & 13 & 21 & 9 \\ 10 & 18 & 1 & 14 & 22 \\ 23 & 6 & 19 & 2 & 15 \end{vmatrix}\,.$$ [*B. U. 1917 Hons. I*]

Add to the elements of the 1st column, the corresponding elements of the remaining four columns (§2·63) and then remove the common factor 65 from the elements of the 1st column of the transformed determinant (§2·62); we then have

$$\Delta = 65 \begin{vmatrix} 1 & 24 & 7 & 20 & 3 \\ 1 & 12 & 25 & 8 & 16 \\ 1 & 5 & 13 & 21 & 9 \\ 1 & 18 & 1 & 14 & 22 \\ 1 & 6 & 19 & 2 & 15 \end{vmatrix}\,.$$

* Subtracting the 4th row from the 5th, the 3rd, from the 4th, and so on, the last four elements of the 1st column of the transformed determinant reduce to zero; hence, expanding it in terms of the elements of the 1st column, we have (§2·6)

$$\triangle = 65 \begin{vmatrix} -12 & 18 & -12 & 13 \\ -7 & -12 & 13 & -7 \\ 13 & -12 & -7 & 13 \\ -12, & 18 & -12 & -7 \end{vmatrix}.$$

[Note how the order of the determinant is reduced.]

We notice that the first three elements of the 1st row are the same as the 1st three, of the last row; therefore subtracting the 4th row from the 1st we have

$$\triangle = 65 \begin{vmatrix} 0 & 0 & 0 & 20 \\ -7 & -12 & 13 & -7 \\ 13 & -12 & -7 & 13 \\ -12 & 18 & -12 & -7 \end{vmatrix}.$$

Now expanding the determinant in terms of elements of the 1st row, we have (§2·6)

$$\triangle = -65.20 \begin{vmatrix} -7 & -12 & 13 \\ 13 & -12 & -7 \\ -12 & 18 & -12 \end{vmatrix}.$$

Now taking out the common factor 6 from the 2nd column and 3 from the last row, we have by § 2·62,

$$\triangle = -65.20.18 \begin{vmatrix} -7 & -2 & 13 \\ 13 & -2 & -7 \\ -4 & 1 & -4 \end{vmatrix}.$$

for the 3! terms
formula not given.

A straightforward application of § 2·6 leads to

$$\triangle = -65.20.18 \left\{ -7 \begin{vmatrix} -2 & -7 \\ 1 & -4 \end{vmatrix} - 13 \begin{vmatrix} -2 & 13 \\ 1 & -4 \end{vmatrix} - 4 \begin{vmatrix} -2 & 13 \\ -2 & -7 \end{vmatrix} \right\}$$

$$= -23400\,[-7(8+7)-13(8-13)-4(14+26)] = 4380000.$$

The student is advised to study this example carefully as it illustrates various devices that we usually employ in reducing a a numerical determinant.

* To subtract the 4th row from the 5th means to subtract the elements of the 4th row from the corresponding elements of the 5th row.

Example. 5. If a, b, c are elements of a determinant of the third order and A, B, C their respective (prepared), minors, simplify

$$(a_1+b_1+c_1)(A_1+B_1+C_1)+(a_2+b_2+c_2)(A_2+B_2+C_2)$$
$$+(a_3+b_3+c_3)(A_3+B_3+C_3). \qquad [B.\ U.\ 1925\ Hons.\ \text{I}]$$

The expression is equal to

$$(a_1A_1+a_2A_2+a_3A_3)+(b_1A_1+b_2A_2+b_3A_3)+(c_1A_1+c_2A_2+c_3A_3)$$
$$+(a_1B_1+a_2B_2+a_3B_3)+(b_1B_1+b_2B_2+b_3B_3)+(c_1B_1+c_2B_2+c_3B_3)$$
$$+(a_1C_1+a_2C_2+a_3C_3)+(b_1C_1+b_2C_2+b_3C_3)+(c_1C_1+c_2C_2+c_3C_3)$$

or $\quad (\Delta+0+0)+(0+\Delta+0)+(0+0+\Delta)=3\Delta.$ (§§ 2·6, 2·61)

Example. 6. Prove that

$$\begin{vmatrix} a+b, & a+2b & a+3b & a+4b \\ a+2b & a+3b & a+4b & a+b \\ a+3b & a+4b & a+b & a+2b \\ a+4b & a+b & a+2b & a+3b \end{vmatrix} = 32(2a+5b)b^3.$$

[*B. U. 1916 Hons.* I]

Add to the 1st column the remaining three columns and then take out the factor $4a+10b$; we then have

$$\Delta = (4a+10b)\times\begin{vmatrix} 1 & a+2b & a+3b & a+4b \\ 1 & a+3b & a+4b & a+b \\ 1 & a+4b & a+b & a+2b \\ 1 & a+b & a+2b & a+3b \end{vmatrix}.$$

Subtracting the 1st row from each of the remaining rows, we have

$$\Delta = (4a+10b)\begin{vmatrix} 1 & a+2b & a+3b & a+4b \\ 0 & b & b & -3b \\ 0 & 2b & -2b & -2b \\ 0 & -b & -b & -b \end{vmatrix}.$$

Now expanding in terms of elements of the 1st column, we have

$$\Delta = (4a+10b)\begin{vmatrix} b & b & -3b \\ 2b & -2b & -2b \\ -b & -b & -b \end{vmatrix}.$$

Now removing the factor b from each of the rows, we get

$$\Delta = 2(2a+5b)b^3\begin{vmatrix} 1 & 1 & -3 \\ 2 & -2 & -2 \\ -1 & -1 & -1 \end{vmatrix}.$$

Add the last row to the first; we then have

$$\triangle = 2(2a+5b)b^3 \begin{vmatrix} 0 & 0 & -4 \\ 2 & -2 & -2 \\ -1 & -1 & -1 \end{vmatrix} = -8(2a+5b)b^3 \begin{vmatrix} 2 & -2 \\ -1 & -1 \end{vmatrix}$$

$$= 32(2a+5b)b^3.$$

A better method will be suggested later on (§7·8).

Example 7. Prove that

$$\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & c^2 & b^2 \\ 1 & c^2 & 0 & a^2 \\ 1 & b^2 & a^2 & 0 \end{vmatrix} = \begin{vmatrix} 0 & a & b & c \\ a & 0 & c & b \\ b & c & 0 & a \\ c & b & a & 0 \end{vmatrix}$$

$$= -(a+b+c)(-a+b+c)(a-b+c)(a+b-c).$$

Multiply rows of the first determinant by abc, a, b, c respectively and then divide the 2nd, the 3rd and the 4th columns of the transformed determinant by bc, ca, ab respectively. The first will thus reduce to the second determinant.

Adding, to the first row, the remaining three of the 2nd determinant, we see that $a+b+c$ is one of its factors. Again multiplying the rows by -1, -1, 1, 1 respectively and adding the last three rows, to the first, we see that $-a+b+c$ is a factor. Similarly, we may find that $a-b+c$, $a+b-c$ are also factors. Since the determinant is of four dimensions, it must be equal to

$$k(a+b+c)(-a+b+c)(a-b+c)(a+b-c)$$

where, on inspection of the coefficient of a^4 on both sides, we find that $-k=+1$ or $k=-1$.

Example 8. Prove that

$$\begin{vmatrix} a^2 & ba^2 & cb^2 \\ ab^2 & bc^2 & ca^2 \\ 2\cos A & 2\cos B & 2\cos C \end{vmatrix} = (a^2+b^2+c^2) \begin{vmatrix} ac & a^2 & bc \\ ab & bc & a^2 \\ 1 & 2\cos B & 2\cos C \end{vmatrix}$$

where a, b, c, A, B, C are elements of a plane triangle.

[*B. U. 1914 Hons. I.*]

Denote the first determinant by $\triangle$. Multiply the last row by abc and then remove the factors a, b, c from the first, second and third columns respectively; we then have

$$\triangle = \begin{vmatrix} c^2 & a^2 & b^2 \\ b^2 & c^2 & a^2 \\ 2bc\cos A & 2ca\cos B & 2ab\cos C \end{vmatrix}.$$

Now add the second and third columns to the first and then substitute for $\cos A$, $\cos B$, $\cos C$ in terms of sides a, b, c. We then have

$$\Delta = \begin{vmatrix} a^2+b^2+c^2 & a^2 & b^2 \\ a^2+b^2+c^2 & c^2 & a^2 \\ a^2+b^2+c^2 & 2ac\cos B & 2ab\cos C \end{vmatrix}$$

$$= (a^2+b^2+c^2)\begin{vmatrix} 1 & a^2 & b^2 \\ 1 & c^2 & a^2 \\ 1 & 2ac\cos B & 2ab\cos C \end{vmatrix}.$$

Multiply the 1st and 2nd rows by ac, ab and remove ac, ab from the 2nd and 3rd columns. Δ then reduces to the desired form.

Example 9. given $f_r(x) = a_r x^3 + 3b_r x^2 + 3c_r x + d_r$ for $r = 1, 2, 3$ prove that

$$\begin{vmatrix} f_1(x) & f_1'(x) & f_1''(x) \\ f_2(x) & f_2'(x) & f_2''(x) \\ f_3(x) & f_3'(x) & f_3''(x) \end{vmatrix} = -18\begin{vmatrix} 1 & -x & x^2 & -x^3 \\ a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix}.$$

[*B. U. 1924 Problems*]

Denote the right hand determinant by Δ. Multiply the first three columns by x^3, $3x^2$, $3x$ respectively and then add them to the fourth column. We then have

$$\Delta = \begin{vmatrix} 1 & -x & x^2 & 0 \\ a_1 & b_1 & c_1 & f_1(x) \\ a_2 & b_2 & c_2 & f_2(x) \\ a_3 & b_3 & c_3 & f_3(x) \end{vmatrix}.$$

Now multiply the first three columns by $3x^2$, $6x$, 3 and then add the first two to the third. Δ then reduces to

$$\tfrac{1}{3}\begin{vmatrix} 1 & -x & 0 & 0 \\ a_1 & b_1 & f_1'(x) & f_1(x) \\ a_2 & b_2 & f_2'(x) & f_2(x) \\ a_3 & b_3 & f_3'(x) & f_3(x) \end{vmatrix}.$$

Now multiply the first two columns of this transformed determinant by $6x$ and 6 respectively and then add the first to the second. We then have

$$\Delta = \tfrac{1}{18}\begin{vmatrix} 1 & 0 & 0 & 0 \\ a_1 & f_1''(x) & f_1'(x) & f_1(x) \\ a_2 & f_2''(x) & f_2'(x) & f_2(x) \\ a_3 & f_3''(x) & f_3'(x) & f_3(x) \end{vmatrix}.$$

which, by expanding in terms of the elements of the first row, is reduced to

$$\frac{1}{8}\begin{vmatrix} f_1''(x) & f_1'(x) & f_1(x) \\ f_2''(x) & f_2'(x) & f_2(x) \\ f_3''(x) & f_3'(x) & f_3(x) \end{vmatrix}.$$

Interchanging the 1st and the 3rd columns, we reduce it to the required form. A more elegant method is suggested later on, (*Example.* 3. §2·71); a third is given in Burnside and Pantons, 'The Theory of Equations,' part II page 53, *Example* 9.

Example 10. Prove that $\begin{vmatrix} \sin^2 x & \sin^2(a-x) & \sin^2(a+x) \\ \sin^2 y & \sin^2(a-y) & \sin^2(a+y) \\ \sin^2 z & \sin^2(a-z) & \sin^2(a+z) \end{vmatrix}$

$= 4\sin^3 a.\cos a.\sin(y-z).\sin(z-x).\sin(x-y).$

[*B.U. 1921 Problems*]

Denote as usual the determinant by $\triangle$ and denote $\tan a$, $\tan x$, $\tan y$, $\tan z$ by t, t_1, t_2, t_3. Taking out factors $\cos^2 x$, $\cos^2 y$, $\cos^2 z$ from the three rows and $\cos^2 a$ from the 2nd and 3rd columns, we have

$$\triangle = \cos^4 a.\cos^2 x.\cos^2 y.\cos^2 z \begin{vmatrix} t_1^2 & (t-t_1)^2 & (t+t_1)^2 \\ t_2^2 & (t-t_2)^2 & (t+t_2)^2 \\ t_3^2 & (t-t_3)^2 & (t+t_3)^2 \end{vmatrix}.$$

Subtract the 2nd column from the 3rd and the 1st from the 2nd and take out factors $4t$ and t from the last two columns. $\triangle$ then reduces to

$$4\cos^2 a.\sin^2 a.\cos^2 x.\cos^2 y.\cos^2 z \begin{vmatrix} t_1^2 & t-2t_1 & t_1 \\ t_2^2 & t-2t_2 & t_2 \\ t_3^2 & t-2t_3 & t_3 \end{vmatrix}.$$

Add twice the 3rd column to the 2nd and then take out the factor t from the 2nd. We then have

$$\triangle = 4\sin^3 a.\cos a.\cos^2 x.\cos^2 y.\cos^2 z \begin{vmatrix} t_1^2 & 1 & t_1 \\ t_2^2 & 1 & t_2 \\ t_3^2 & 1 & t_3 \end{vmatrix}$$

$$= 4\sin^3 a.\cos a.\cos^2 x.\cos^2 y.\cos^2 z\,(t_2-t_3)(t_3-t_1)(t_1-t_2)$$

$$= 4\sin^3 a.\cos a.\sin(y-z).\sin(z-x).\sin(x-y).$$

[see *Example* 1, §2·411]

A different method is suggested further on (*Example* 2. §2·71).

Examples 11. Show that $\begin{vmatrix} ax-by-cz & ay+bx & cx+az \\ ay+bx & by-cz-ax & bz+cy \\ cx+az & bz+cy & cz-ax-by \end{vmatrix}$

$= (a^2+b^2+c^2)(x^2+y^2+z^2)(ax+by+cz).$

[*B. U. 1923 Problems.*]

Denote the determinant by Δ. Multiply the rows by a, b, c and then add to the 1st, the remaining rows; we have

$$a\Delta = \begin{vmatrix} x(a^2+b^2+c^2) & y(a^2+b^2+c^2) & z(a^2+b^2+c^2) \\ ay+bx & by-cz-ax & bz+cy \\ cx+az & bz+cy & cz-ax-by \end{vmatrix}$$

$$= (a^2+b^2+c^2)\begin{vmatrix} x & y & z \\ ay+bx & by-cz-ax & bz+cy \\ cx+az & bz+cy & cz-ax-by \end{vmatrix}.$$

Now multiply columns of the resulting determinant by x, y, z respectively and then add the 2nd and the 3rd to the 1st column. We then have

$$ax\,\Delta = \begin{vmatrix} (x^2+y^2+z^2) & y & z \\ (x^2+y^2+z^2)b & by-cz-ax & bz+cy \\ (x^2+y^2+z^2)c & bz+cy & cz-ax-by \end{vmatrix}(a^2+b^2+c^2)$$

$$= (a^2+b^2+c^2)(x^2+y^2+z^2)\begin{vmatrix} 1 & y & z \\ b & by-cz-ax & bz+cy \\ c & bz+cy & cz-ax-by \end{vmatrix}.$$

Now multiply the 1st column by y and then subtract it from the 2nd. Again multiply the 1st column by z and then subtract it from the 3rd. Expanding in terms of the elements of the 1st row, we have.

$$ax\,\Delta = \begin{vmatrix} -cz-ax & cy \\ bz & -ax-by \end{vmatrix}(a^2+b^2+c^2)(x^2+y^2+z^2)$$

$$= ax(a^2+b^2+c^2)(x^2+y^2+z^2)(ax+by+cz)$$

$$\therefore \Delta = (a^2+b^2+c^2)(x^2+y^2+z^2)(ax+by+cz).$$

Examples II. d.

Evaluate the following determinants:

1. $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix}$; 2. $\begin{vmatrix} 2 & 3 & 4 \\ 2 & 6 & 12 \\ 1 & 4 & 9 \end{vmatrix}$; 3. $\begin{vmatrix} 3 & 2 & 1 \\ 2 & 5 & 3 \\ 5 & 13 & 9 \end{vmatrix}$; 4. $\begin{vmatrix} 1 & 7 & 13 \\ 1 & 2 & 3 \\ 2 & 6 & 12 \end{vmatrix}$.

5. $\begin{vmatrix} 3 & 1 & 2 \\ 6 & 2 & 5 \\ 14 & 4 & 13 \end{vmatrix}$; 6. $\begin{vmatrix} 5 & 5 & 6 & 4 \\ 2 & 3 & 3 & 2 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{vmatrix}$; 7. $\begin{vmatrix} 2 & 9 & 25 & 55 \\ 2 & 7 & 13 & 30 \\ 2 & 5 & 9 & 14 \\ 2 & 3 & 4 & 5 \end{vmatrix}$.

8. $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 6 \\ 5 & 6 & 7 & 8 \\ 7 & 8 & 9 & 10 \end{vmatrix}$; 9. $\begin{vmatrix} -1 & 0 & 1 & 2 \\ -1 & -1 & 0 & 2 \\ -1 & -2 & -2 & 0 \\ -1 & -3 & -5 & -5 \end{vmatrix}$; 10. $\begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \\ 1 & 4 & 6 & 4 \end{vmatrix}$;

11. $\begin{vmatrix} a & b & c \\ a & a+b & a+b+c \\ a & 2a+b & 3a+2b+c \end{vmatrix}$; 12. $\begin{vmatrix} a_1-d_1 & a_1 & a_1+d_1 \\ a_2-d_2 & a_2 & a_2+d_2 \\ a_3-d_3 & a_3 & a_3+d_3 \end{vmatrix}=0$;

Prove that

13. $\begin{vmatrix} 1 & 5 & 20 & 75 \\ 5 & 20 & 75 & 275 \\ 20 & 75 & 275 & 1001 \\ 75 & 275 & 1001 & 3640 \end{vmatrix}=5$;

14. $\begin{vmatrix} 1 & 5 & 20 & 75 & 275 \\ 5 & 20 & 75 & 275 & 1001 \\ 20 & 75 & 275 & 1001 & 3640 \\ 75 & 275 & 1001 & 3640 & 13260 \\ 275 & 1001 & 3640 & 13260 & 48450 \end{vmatrix}=1$.

The two examples 13,14 are extracted from a paper by M. Bhimasena Rao and M. Venkatarama Ayyar published in I. M. S. Journal, vol. XVI no. 11.

15. $\begin{vmatrix} {}_3C_0 & {}_3C_1 & {}_3C_2 & {}_3C_3 \\ {}_3C_1 & {}_3C_2 & {}_3C_3 & {}_3C_0 \\ {}_3C_2 & {}_3C_3 & {}_3C_0 & {}_3C_1 \\ {}_3C_3 & {}_3C_0 & {}_3C_1 & {}_3C_2 \end{vmatrix}=0$;

16. $\begin{vmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ -a & -b & -c & 1 \end{vmatrix}=1+a^2+b^2+c^2$;

17. $\begin{vmatrix} \cos x & \cos 2x & \cos 3x \\ \cos 4x & \cos 5x & \cos 6x \\ \cos 6x & \cos 7x & \cos 8x \end{vmatrix}=0$;

18. $\begin{vmatrix} \cos(a_1-x) & \cos a_1 & \cos(a_1+x) \\ \cos(a_2-x) & \cos a_2 & \cos(a_2+x) \\ \cos(a_3-x) & \cos a_3 & \cos(a_3+x) \end{vmatrix}=0$;

19. $\begin{vmatrix} m_2+m_3 & m_1(1-m_2m_3) & 1-m_2m_3 \\ m_3+m_1 & m_2(1-m_1m_3) & 1-m_1m_3 \\ m_1+m_2 & m_3(1-m_1m_2) & 1-m_1m_2 \end{vmatrix}$

$=(m_2-m_3)(m_3-m_1)(m_1-m_2)(m_1m_2m_3-m_1-m_2-m_3)$;

20. $$\begin{vmatrix} m_1m_3+m_1m_2 & 1-m_2m_3 & m_1(1-m_2m_3) \\ m_2m_3+m_2m_1 & 1-m_1m_3 & m_2(1-m_1m_3) \\ m_3m_1+m_3m_2 & 1-m_1m_2 & m_3(1-m_1m_2) \end{vmatrix}$$
$$= (m_2m_3+m_3m_1+m_1m_2-1)(m_2-m_3)(m_3-m_1)(m_1-m_2);$$

21. $$\begin{vmatrix} 1 & m_1^2 & m_1^3 \\ 1 & m_2^2 & m_2^3 \\ 1 & m_3^2 & m_3^3 \end{vmatrix}$$
$$= (m_2m_3+m_3m_1+m_1m_2)(m_2-m_3)(m_3-m_1)(m_1-m_2);$$

22. $$\begin{vmatrix} 1 & b' & a' & 1 \\ b & 1+bb' & a'b & b-b' \\ a & b'a & 1+a'a & -a'+a \\ 1 & -b+b' & -a+a' & aa'+bb'+cc'+1 \end{vmatrix} = cc';$$

23. $$\begin{vmatrix} a & b\cos C & c\cos B \\ b & c\cos A & a\cos C \\ c & a\cos B & b\cos A \end{vmatrix} = 0;$$

where a, b, c A, B, C are elements of a plane triangle.

24. $$\begin{vmatrix} m_1N_1-n_1M_1 & n_1L_1-l_1N_1 & l_1M_1-m_1L_1 \\ m_2N_2-n_2M_2 & n_2L_2-l_2N_2 & l_2M_2-m_2L_2 \\ m_3N_3-n_3M_3 & n_3L_3-l_3N_3 & l_3M_3-m_3L_3 \end{vmatrix} = 0;$$

where $L_1, M_1, N_1, L_2, \dots$, are cofactors of $l_1, m_1, n_1, l_2, \dots$, in

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}.$$

25. $$(la_1+mb_1+nc_1)(l'A_1+m'B_1+n'C_1)$$
$$+(la_2+mb_2+nc_2)(l'A_2+m'B_2+n'C_2)$$
$$+(la_3+mb_3+nc_3)(l'A_3+m'B_3+n'C_3) = (ll'+mm'+nn')\Delta$$

where $\Delta = [a_1b_2c_3]$ and $A_1, B_1, C_1 \dots$ are the cofactors of $a_1, b_1, c_1 \dots$.

26. $$\begin{vmatrix} -a & b+c & b+c \\ c+a & -b & c+a \\ a+b & a+b & -c \end{vmatrix} = (a+b+c)^3;$$

27. Solve the equation $$\begin{vmatrix} x+a & x+b & x+c \\ x+c & x+a & x+b \\ x+b & x+c & x+a \end{vmatrix} = 0;$$

28. If the vertices A, B, C of a triangle are given by

$$(x_1/z_1, y_1/z_1),\ (x_2/z_2, y_2/z_2),\ (x_3/z_3, y_3/z_3),$$

then the equations to BC, CA, AB are

$xX_1+yY_1+Z_1=0$, $xX_2+yY_2+Z_2=0$ and $xX_3+yY_3+Z_3=0$ where X_1, Y_1,...are cofactors of $x_1, y_1, \ldots$ in $[x_1\ y_2\ z_3]$.

29. In the notation of example 28, prove that the equation of the line joining A to D which divides BC in the ratio of $l : m$ is given by

$$x(lz_2X_2-mz_3X_3)+y(lz_2Y_2-mz_3Y_3)+(lz_2Z_2-mz_3Z_3)=0.$$

30. Prove that the vertices of a triangle ABC, whose sides are given by $a_rx+b_ry+c_r=0$ $(r=1, 2, 3,)$ are

$$(A_1/C_1,\ B_1/C_1),\ (A_2/C_2, B_2/C_2),\ (A_3/C_3,\ B_3/C_3)$$

where A_1, B_1, ... are cofactors of a_1, b_1, ... in $[a_1\ b_2\ c_3]$.

31. In the notation of example 30, show that the line joining A to D which divides BC in the ratio of $l : m$ is given by

$$x[la_2C_2-ma_3C_3]+y[lb_2C_2-mb_3C_3]+[lc_2C_2-mc_3C_3]=0.$$

This example should be attempted after reading §2·71, *Example* 1.

32 If A, B, C, D are given by $(x_r/z_r, y_r/z_r)$ where $r=1, 2, 3, 4$ then DB divides AC in the ratio of $l : m$ where

$$\begin{vmatrix} x_2 & x_4 \\ lz_1 & z_2 \\ y_2 & y_4 \\ z_2 & z_4 \end{vmatrix} = \begin{vmatrix} x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{vmatrix} = mz_3 \begin{vmatrix} x_4 & y_4 & z_4 \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}.$$

2·7. Multiplication of Determinants. Prove that

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2 \\ a_3 & b_3 & c_3 & d_3 & e_3 & f_3 \\ 0 & 0 & 0 & d_4 & e_4 & f_4 \\ 0 & 0 & 0 & d_5 & e_5 & f_5 \\ 0 & 0 & 0 & d_6 & e_6 & f_6 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} d_4 & e_4 & f_4 \\ d_5 & e_5 & f_5 \\ d_6 & e_6 & f_6 \end{vmatrix}.$$

We shall denote the three determinants by Δ, Δ_1 and Δ_2 resqectively. Then it is clear that the product of a term P of Δ_1 and a term Q of Δ_2 is a term of Δ; for $P.Q$ is formed by taking one and only one element from each row and one and only one from each column of Δ. Further the number of inversions in $P.Q$ is equal to the sum of the number of inversions in P and the number of those in Q. Hence $P.Q$ is a term of Δ with its proper sign. Again any other way of forming a term of Δ involves the selection of a zero element; hence all other terms of Δ are zero.

$$\therefore \quad \Delta = \Delta_1 \times \Delta_2.$$

The result is obviously capable of extension.

2·71. We are now in a position to prove, what is known as, the Multiplication Theorem for determinants:

Prove that $[a_1\ b_2\ c_3] \times [p_1\ q_2\ r_3]$

$$= \begin{vmatrix} a_1p_1+a_2q_1+a_3r_1 & b_1p_1+b_2q_1+b_3r_1 & c_1p_1+c_2q_1+c_3r_1 \\ a_1p_2+a_2q_2+a_3r_2 & b_1p_2+b_2q_2+b_3r_2 & c_1p_2+c_2q_2+c_3r_2 \\ a_1p_3+a_2q_3+a_3r_3 & b_1p_3+b_2q_3+b_3r_3 & c_1p_3+c_2q_3+c_3r_3 \end{vmatrix}$$

Consider the determinant

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 & -1 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & -1 & 0 \\ a_3 & b_3 & c_3 & 0 & 0 & -1 \\ 0 & 0 & 0 & p_1 & q_1 & r_1 \\ 0 & 0 & 0 & p_2 & q_2 & r_2 \\ 0 & 0 & 0 & p_3 & q_3 & r_3 \end{vmatrix}.$$

Multiply the first three rows by p_1, q_1, r_1 respectively and then add them all to the fourth. Multiply the same rows of the transformed determinant by p_2, q_2, r_2 respectively and then add them to the fifth row. Lastly, multiply by p_3, q_3, r_3 and then add them to the last row. The determinant then reduces to

$$\begin{vmatrix} a_1 & b_1 & c_1 & -1 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & -1 & 0 \\ a_3 & b_3 & c_3 & 0 & 0 & -1 \\ a_1p_1+a_2q_1+a_3r_1 & b_1p_1+b_2q_1+b_3r_1 & c_1p_1+c_2q_1+c_3r_1 & 0 & 0 & 0 \\ a_1p_2+a_2q_2+a_3r_2 & b_1p_2+b_2q_2+b_3r_2 & c_1p_2+c_2q_2+c_3r_2 & 0 & 0 & 0 \\ a_1p_3+a_2q_3+a_3r_3 & b_1p_3+b_2q_3+b_3r_3 & c_1p_3+c_2q_3+c_3r_3 & 0 & 0 & 0 \end{vmatrix}$$

Now interchange the first and the fourth rows, the second and the fifth, and the third and the sixth, of the latter determinant. It will then transform into a determinant of the same form as that of §2.7. Hence it is equal to the product of

$$(-1)^3 \begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix}$$

and $\begin{vmatrix} a_1p_1+a_2q_1+a_3r_1 & b_1p_1+b_2q_1+b_3r_1 & c_1p_1+c_2q_1+c_3r_1 \\ a_1p_2+a_2q_2+a_3r_2 & b_1p_2+b_2q_2+b_3r_2 & c_1p_2+c_2q_2+c_3r_2 \\ a_1p_3+a_2q_3+a_3r_3 & b_1p_3+b_2q_3+b_3r_3 & c_1p_3+c_2q_3+c_3r_3 \end{vmatrix}$.

Now observe that Δ is by §2·7, equal to $[a_1\ b_2\ c_3] \times [p_1\ q_2\ r_3]$

$$\begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = (-1)^3;$$

the required result becomes quite obvious. This theorem may be extended to determinants of higher orders. We further observe that by interchanging rows and columns of each of the determinants $[a_1\, b_2\, c_3]$ and $[p_1\, q_2\, r_3]$ and applying the above rule for multiplying determinants, we see that the product determinant may be exhibited in four distinct ways. Thus $[a_1\, b_2] \times [p_1\, q_2]$ ...

$$= \begin{vmatrix} a_1p_1+b_1q_1 & a_1p_2+b_1q_2 \\ a_2p_1+b_2q_1 & a_2p_2+b_2q_2 \end{vmatrix} = \begin{vmatrix} a_1p_1+a_2q_1 & a_1p_2+a_2q_2 \\ b_1p_1+b_2q_1 & b_1p_2+b_2q_2 \end{vmatrix}$$

$$= \begin{vmatrix} a_1p_1+b_1p_2 & a_1q_1+b_1q_2 \\ a_2p_1+b_2p_2 & a_2q_1+b_2q_2 \end{vmatrix} = \begin{vmatrix} a_1p_1+a_2p_2 & a_1q_1+a_2q_2 \\ b_1p_1+b_2p_2 & b_1q_1+b_2q_2 \end{vmatrix}.$$

The following examples illustrate some important applications of the above theorem.

Example. 1. Prove that (i) $[A_1\, B_2\, C_3] = [a_1\, b_2\, c_3]^2$;

(ii) $\begin{vmatrix} B_2 & C_2 \\ B_3 & C_3 \end{vmatrix} = a_1\Delta$; $\begin{vmatrix} A_2 & C_2 \\ A_3 & C_3 \end{vmatrix} = b_1\Delta$; etc.

where A_1, B_1, C_1, ... are cofactors of a_1, b_1, c_1 ... in $[a_1\, b_2\, c_3]$ or Δ.

(i) Now by §2·71, we have

$$[a_1\, b_2\, c_3] \times [A_1\, B_2\, C_3]$$

$$= \begin{vmatrix} a_1A_1+b_1B_1+c_1C_1 & a_1A_2+b_1B_2+c_1C_2 & a_1A_3+b_1B_3+c_1C_3 \\ a_2A_1+b_2B_1+c_2C_1 & a_2A_2+b_2B_2+c_2C_2 & a_2A_3+b_2B_3+c_2C_3 \\ a_3A_1+b_3B_1+c_3C_1 & a_3A_2+b_3B_2+c_3C_2 & a_3A_3+b_3B_3+c_3C_3 \end{vmatrix}$$

which, in virtue of §§2·6,2·601, reduces to

$$\begin{vmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{vmatrix} \text{ or } \Delta^3 \text{ where } \Delta \equiv [a_1\, b_2\, c_3];$$

and hence,

$$[A_1\, B_2\, C_3] = [a_1\, b_2\, c_3]^2.$$

(ii) To prove the other part, we consider the product

$$\begin{vmatrix} 1 & 0 & 0 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} \times \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

Applying the rule of § 2·71 and using relations of §§2·6, 2·601, we reduce the product to

$$\begin{vmatrix} a_1 & 0 & 0 \\ a_2 & \Delta & 0 \\ a_3 & 0 & \Delta \end{vmatrix} \quad or \quad a_1\Delta^2.$$

Now $$\begin{vmatrix} 1 & 0 & 0 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = \begin{vmatrix} B_2 & C_2 \\ B_3 & C_3 \end{vmatrix}.$$

and hence $$\begin{vmatrix} B_2 & C_2 \\ B_3 & C_3 \end{vmatrix} = a_1\Delta.$$

Other results may be similarly established. Similar results may be proved for determinants of higher orders. A determinant whose elements are the cofactors of another determinant, is called a *reciprocal determinant*. [Read vol. II, page 41 , of Burnside and Pantons' Theory of Equations.]

Example 2. Give an alternative proof of *Example* 10, §2·64.

In the notation of that example, the determinant

$$\Delta = \cos^2x.\cos^2y.\cos^2z.\cos^4a \begin{vmatrix} t_1^2 & (t-t_1)^2 & (t+t_1)^2 \\ t_2^2 & (t-t_2)^2 & (t+t_2)^2 \\ t_3^2 & (t-t_3)^2 & (t+t_3)^2 \end{vmatrix}.$$

Now it is easily seen that the latter determinant is equal to

$$\begin{vmatrix} t_1^2 & 1 & t_1 \\ t_2^2 & 1 & t_2 \\ t_3^2 & 1 & t_3 \end{vmatrix} \times \begin{vmatrix} 1 & 0 & 0 \\ 1 & t^2 & -2t \\ 1 & t^2 & 2t \end{vmatrix};$$

but these two determinants are respectively equal to

$$(t_2-t_3)(t_3-t_1)(t_1-t_2) \quad \text{and} \quad 4t^3,$$

and thus we get the required result

Example 3. Prove *Example* 9, §2·64, by applying the multiplication theorem.

Now $$\begin{vmatrix} 1 & -x & x^2 & -x^3 \\ a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix} \times \begin{vmatrix} x^3 & 3x^2 & 3x & 1 \\ 3x^2 & 6x & 3 & 0 \\ 6x & 6 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix}$$

is, by §2·71, equal to

$$\begin{vmatrix} 0 & 0 & 0 & 1 \\ f_1(x) & f_1'(x) & f_1''(x) & a_1 \\ f_2(x) & f_2'(x) & f_2''(x) & a_2 \\ f_3(x) & f_3'(x) & f_3''(x) & a_3 \end{vmatrix} \text{ or } - \begin{vmatrix} f_1(x) & f_1'(x) & f_1''(x) \\ f_2(x) & f_2'(x) & f_2''(x) \\ f_3(x) & f_3'(x) & f_3''(x) \end{vmatrix};$$

and $$\begin{vmatrix} x^3 & 3x^2 & 3x & 1 \\ 3x^2 & 6x & 3 & 0 \\ 6x & 6 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} = 18.$$

Thus the required result is obvious.

Example 4. Find the product of the two determinants

$$\begin{vmatrix} a+ib & c+id \\ -c+id & a-ib \end{vmatrix} \times \begin{vmatrix} a_1-ib_1 & c_1-id_1 \\ -c_1-id_1 & a_1+ib_1 \end{vmatrix}$$

where $i = \sqrt{(-1)}$. Hence prove Euler's Theorem that the product of two sums each of four squares can be expressed as the sum of four squares. [*B. U. 1924 Hons. I.*]

The two determinants are respectively equal to $a^2+b^2+c^2+d^2$ and $a_1^2+b_1^2+c_1^2+d_1^2$; and their product is equal to

$$\begin{vmatrix} P & Q \\ R & S \end{vmatrix}$$

where P, Q, R, S are respectively the same as

$(a+ib)(a_1-ib_1)+(c+id)(c_1-id_1)$, $(a_1+ib)(c_1-id_1)+(c+id)(a_1+ib_1)$,

$(-c+id)(a_1-ib_1)+(a-ib)(c_1-id_1)$, $(-c+id)(-c_1-id_1)+(a-ib)(a_1+ib_1)$.

On inspection, we find that this determinant is of the form

$$\begin{vmatrix} A+iB & C+iD \\ -C+iD & A-iB \end{vmatrix}$$ which is equal to $A^2+B^2+C^2+D^2$.

It is evident from this example that the product of any number of factors each of which is a sum of four squares can be exhibited as the sum of four squares.

We give below some geometrical applications.

Examples 5. D, E, F are the vertices of a triangle inscribed in another $\triangle ABC$. Show that

$$4R.\ \triangle DEF = BD.\ CE.\ AF + DC.\ EA.\ FB.$$

Let the coordinates of A, B, C referred to a pair of rectangular axes be (x_1, y_1), (x_2, y_2) and (x_3, y_3) and let D, E, F divide the sides BC, CA, AB, respectively in the ratios of $l_1 : m_1$, $l_2 : m_2$ and $l_3 : m_3$. The coordinates of D, E, F are then respectively given by

$$(l_1x_3+m_1x_2)/(l_1+m_1),\ (l_1y_3+m_1y_2)/(l_1+m_1);$$

$$(l_2x_1+m_2x_3)/(l_2+m_2),\ (l_2y_1+m_2y_3)/(l_2+m_2);$$

and $$(l_3x_2+m_3x_1)/(l_3+m_3),\ (l_3y_2+m_3y_1)/(l_3+m_3);$$

$$\therefore \Delta DEF = \tfrac{1}{2}\begin{vmatrix} (l_1x_3+m_1x_2)/(l_1+m_1) & (l_1y_3+m_1y_2)/(l_1+m_1) & 1 \\ (l_2x_1+m_2x_3)/(l_2+m_2) & (l_2y_1+m_2y_3)/(l_2+m_2) & 1 \\ (l_3x_2+m_3x_1)/(l_3+m_3) & (l_3y_2+m_3y_1)/(l_3+m_3) & 1 \end{vmatrix}$$

$$= \frac{1}{2(l_1+m_1)(l_2+m_2)(l_3+m_3)} \begin{vmatrix} l_1x_3+m_1x_2 & l_1y_3+m_1y_2 & l_1+m_1 \\ l_2x_1+m_2x_3 & l_2y_1+m_2y_3 & l_2+m_2 \\ l_3x_2+m_3x_1 & l_3y_2+m_3y_1 & l_3+m_3 \end{vmatrix}$$

$$= \frac{1}{2(l_1+m_1)(l_2+m_2)(l_3+m_3)} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \times \begin{vmatrix} 0 & m_1 & l_1 \\ l_2 & 0 & m_2 \\ m_3 & l_3 & 0 \end{vmatrix}$$

$$= \frac{(l_1l_2l_3+m_1m_2m_3)}{(l_1+m_1)(l_2+m_2)(l_3+m_3)} \Delta ABC$$

which, by substituting $l_1/m_1 = BD/DC$, etc. reduces to

$$\frac{(BD.\ CE.\ AF+DC.\ EA.FB)}{BC.\ CA.\ AB} \Delta ABC.$$

Now $$BC.\ CA.\ AB = 4R.\Delta ABC,$$

and thus we have the required result immediately.

Example 6. Prove that if

$$x^2=(b-c)^2+(b'-c')^2, y^2=(c-a)^2+(c'-a')^2, z^2=(a-b)^2+(a'-b')^2,$$

$$4.\begin{vmatrix} 1 & a & a' \\ 1 & b & b' \\ 1 & c & c' \end{vmatrix}^2 = (x+y+z)(y+z-x)(z+x-y)(x+y-z).$$

Interpret the result geometrically. [*B. U. 1922 Hons.* 1]

It is easily seen that

$$4.\begin{vmatrix} 1 & a & a' \\ 1 & b & b' \\ 1 & c & c' \end{vmatrix} = 4\begin{vmatrix} 1 & 0 & 0 & 0 \\ a^2+a'^2 & a & a' & 1 \\ b^2+b'^2 & b & b' & 1 \\ c^2+c'^2 & c & c' & 1 \end{vmatrix} = -\begin{vmatrix} 0 & 0 & 0 & 1 \\ 1 & -2a & -2a' & a^2+a'^2 \\ 1 & -2b & -2b' & b^2+b'^2 \\ 1 & -2c & -2c' & c^2+c'^2 \end{vmatrix};$$

$$\therefore 4\begin{vmatrix} 1 & a & a' \\ 1 & b & b' \\ 1 & c & c' \end{vmatrix}^2 = -\begin{vmatrix} 1 & 0 & 0 & 0 \\ a^2+a'^2 & a & a' & 1 \\ b^2+b'^2 & b & b' & 1 \\ c^2+c'^2 & c & c' & 1 \end{vmatrix} \times \begin{vmatrix} 0 & 0 & 0 & 1 \\ 1 & -2a & -2a' & a^2+a'^2 \\ 1 & -2b & -2b' & b^2+b'^2 \\ 1 & -2c & -2c' & c^2+c'^2 \end{vmatrix}$$

$$= -\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & z^2 & y^2 \\ 1 & z^2 & 0 & x^2 \\ 1 & y^2 & x^2 & 0 \end{vmatrix} = (x+y+z)(y+z-x)(z+x-y)(x+y-z).$$

(See illustrative *Example* 7. §2·64)

If $(a,a')(b,b')(c,c')$ are coordinates of the vertices of a triangle referred to rectangular axes, then x, y, z denote lengths of sides and

$$\begin{vmatrix} 1 & a & a' \\ 1 & b & b' \\ 1 & c & c' \end{vmatrix}$$

denotes twice the area of the triangle. Hence the geometrical meaning of the above result is that the area of a triangle whose sides are x, y, z is $\frac{1}{4}[(x+y+z)(-x+y+z)(x-y+z)(x+y-z)]^{\frac{1}{2}}$.

Example 7. ABC is a spherical triangle and P, Q, R are three points on a great circle. If cosines of arcs PA, PB, PC be denoted by a_1, b_1, c_1; those of QA, QB, QC, by a_2, b_2, c_2; and thoseof RA, RB, RC, by a_3, b_3, c_3, prove that

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

[*c. f. B. U. 1916 Problems*]

Choose the radius of the sphere for the unit of length, and then refer the sphere to a set of rectangular axes passing through O, the centre of the sphere. If we denote the direction cosines of OA, OB, OC by l_r, m_r, n_r $[r = 1, 2, 3]$ and tnose of OP, OQ, OR by l_r', m_r', n_r' $[r=1, 2, 3]$; it is then clear that coordinates of P, Q, R are (l_r, m_r, n_r) $[r=1, 2, 3]$. Since P, Q, R lie on a great circle, points P, Q, R lie on a plane through the origin; and hence

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = 0 .$$

Now remembering that $l_1l_1' + m_1m_1' + n_1n_1' = a_1$, etc. we have

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} \times \begin{vmatrix} l_1' & m_1' & n_1' \\ l_2' & m_2' & n_2' \\ l_3' & m_3' & n_3' \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

The required result is now quite obvious.

Examples II. e.

Prove that:

1. $$\begin{vmatrix} a_1 & a_2 & a_3 \\ la_1+b_1 & la_2+b_2 & la_3+b_3 \\ ma_1+nb_1+c_1 & ma_2+nb_2+c_2 & ma_3+nb_3+c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

2. $$\begin{vmatrix} (l_1+m_1)^2 & (l_1+m_2)^2 & (l_1+m_3)^2 \\ (l_2+m_1)^2 & (l_2+m_2)^2 & (l_2+m_3)^2 \\ (l_3+m_1)^2 & (l_3+m_2)^2 & (l_3+m_3)^2 \end{vmatrix}$$
$$= -2(m_2-m_3)(m_3-m_1)(m_1-m_2)\,(l_2-l_3)(l_3-l_1)(l_1-l_2)\,.$$

Multiply $\begin{vmatrix} m_1^2 & m_1 & 1 \\ m_2^2 & m_2 & 1 \\ m_3^2 & m_3 & 1 \end{vmatrix}$ and $\begin{vmatrix} 1 & 2l_1 & l_1^2 \\ 1 & 2l_2 & l_2^2 \\ 1 & 2l_3 & l_3^2 \end{vmatrix}$.

3. $$\begin{vmatrix} (x+p)^3 & (y+p)^3 & (z+p)^3 & (w+p)^3 \\ (x+q)^2 & (y+q)^2 & (z+q)^2 & (w+q)^2 \\ x+r & y+r & z+r & w+r \\ 1 & 1 & 1 & 1 \end{vmatrix}$$
$$= (x-y)(x-z)(x-w)(y-z)(y-w)(z-w).$$

4. $$\begin{vmatrix} x^4-p^4 & y^4-p^4 & z^4-p^4 & w^4-p^4 \\ x^3-p^3 & y^3-p^3 & z^3-p^3 & w^3-p^3 \\ x^2-p^2 & y^2-p^2 & z^2-p^2 & w^2-p^2 \\ x-p & y-p & z-p & w-p \end{vmatrix}$$
$$= (x-p)(y-p)(z-p)(w-p)(x-y)(x-z)(x-w)(y-z)(y-w)(z-w),$$

the last six factors being the cofactors.

5. $$\begin{vmatrix} s_4 & s_3 & s_2 \\ s_3 & s_2 & s_1 \\ s_2 & s_1 & s_0 \end{vmatrix} = (x-y)^2(x-z)^2(y-z)^2$$

where $s_r = x^r + y^r + z^r$.

References.

Burnside and Panton. Theory of Equations, Vol. II, Chap. XIII, Note A, Pp. 287–288, gives a brief historical account of the subject.

Scott and Mathews- Theory of Determinants. Chapters I, II, III, IV, V. In this book the theory is based on that of Alternate Numbers. The method is more powerful and at the same time, more natural than the one adopted in the book. Any student who has grasped the Fundamental Laws of Algebra should easily follow it.

Miller. Historical Introduction to Mathematical Literature. Chapter VI, §39, contains historical information.

Examples. II. f.

1. In the determinant $[a_1 b_2 c_3 \ldots e_r \ldots l_{n-1} m_n]$ find the signs of terms (1) $a_1 b_2 c_3 \ldots e_r f_n g_{n-1} \ldots l_{r+2} m_{r+1}$ (2) $a_2 b_4 c_6 \ldots k_5 l_3 m_1$, and (3) $a_2 b_1 c_4 l_3 \ldots$

Prove the following results:

2. $$\begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ -a & -b & -c & d \end{vmatrix} = a+b+c+d.$$

3. $$[a_1\ b_2\ c_3\ d_4] = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ 0 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ 0 & b_4 & c_4 & d_4 \end{vmatrix} + \begin{vmatrix} 0 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ 0 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}.$$

4. $$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = - \begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \\ x_4-x_1 & y_4-y_1 & z_4-z_1 \end{vmatrix}.$$

5. $$a_1^3[a_1\ b_2\ c_3\ d_4] = \begin{vmatrix} a_1b_2-a_2b_1 & a_1c_2-a_2c_1 & a_1d_2-a_2d_1 \\ a_1b_3-a_3b_1 & a_1c_3-a_3c_1 & a_1d_3-a_3d_1 \\ a_1b_4-a_4b_1 & a_1c_4-a_4c_1 & a_1d_4-a_4d_1 \end{vmatrix}.$$

6. $$[a_1\ b_2\ c_3\ d_4] = \begin{vmatrix} 0 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} + a_1 \begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix}.$$

7. $$\begin{vmatrix} 1 & 0 & x_1 & y_1 \\ 0 & 1 & x_2 & y_2 \\ 1 & 0 & x_3 & y_3 \\ 0 & 1 & x_4 & y_4 \end{vmatrix} = \begin{vmatrix} x_3-x_1 & y_3-y_1 \\ x_4-x_2 & y_4-y_2 \end{vmatrix}.$$

8. $$\begin{vmatrix} a^3 & b^3 & c^3 \\ \cos A & \cos B & \cos C \\ \sin A & \sin B & \sin C \end{vmatrix} = 0, \quad \approx \begin{vmatrix} \sec A & \sec B & \sec C \\ \operatorname{cosec} A & \operatorname{cosec} B & \operatorname{cosec} C \\ \cos A & \cos B & \cos C \end{vmatrix} = 0$$

where a, b, c, A, B, C are elements of a plane triangle.

9. $$\begin{vmatrix} a_1 & b_1 & 0 & 0 & 0 & 0 \\ a_2 & b_2 & 0 & 0 & 0 & 0 \\ a_3 & b_3 & c_3 & d_3 & 0 & 0 \\ a_4 & b_4 & c_4 & d_4 & 0 & 0 \\ a_5 & b_5 & c_5 & d_5 & e_5 & f_5 \\ a_6 & b_6 & c_6 & d_6 & e_6 & f_6 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} c_3 & d_3 \\ c_4 & d_4 \end{vmatrix} \times \begin{vmatrix} e_5 & f_5 \\ e_6 & f_6 \end{vmatrix}.$$

10. $$\begin{vmatrix} b+c+na & (n-1)a & (n-1)a \\ (n-1)b & c+a+nb & (n-1)b \\ (n-1)c & (n-1)c & a+b+nc \end{vmatrix} = n(a+b+c)^3.$$

11. $\begin{vmatrix} -bc & c^2+bc & b^2+bc \\ c^2+ac & -ac & a^2+ac \\ b^2+ab & a^2+ab & -ab \end{vmatrix} = abc(a+b+c)^3.$

12. $\begin{vmatrix} (a+b)(a+c) & a^2 & a^2 \\ b^2 & (b+c)(b+a) & b^2 \\ c^2 & c^2 & (c+a)(c+b) \end{vmatrix}$

$= (bc+ca+ab)^2(a^2+b^2+c^2+bc+ca+ab).$

13. $\begin{vmatrix} -bc & b^2+bc & c^2+cb \\ a^2+ac & -ac & c^2+ca \\ a^2+ab & b^2+ab & -ab \end{vmatrix} = (bc+ca+ab)^3.$

14. Solve the equation $\begin{vmatrix} x & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{vmatrix} = 0.$

15. Prove that each of the four determinants

$$\begin{vmatrix} 0 & a^2l^2 & b^2m^2 & c^2n^2 \\ a^2l^2 & 0 & 1 & 1 \\ b^2m^2 & 1 & 0 & 1 \\ c^2n^2 & 1 & 1 & 0 \end{vmatrix}, \quad \begin{vmatrix} 0 & a^2 & b^2 & c^2 \\ a^2 & 0 & n^2 & m^2 \\ b^2 & n^2 & 0 & l^2 \\ c^2 & m^2 & l^2 & 0 \end{vmatrix},$$

$$\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & c^2n^2 & b^2m^2 \\ 1 & c^2n^2 & 0 & a^2l^2 \\ 1 & b^2m^2 & a^2l^2 & 0 \end{vmatrix}, \quad \begin{vmatrix} 0 & l^2 & m^2 & n^2 \\ l^2 & 0 & c^2 & b^2 \\ m^2 & c^2 & 0 & a^2 \\ n^2 & b^2 & a^2 & 0 \end{vmatrix},$$

is equal to

$$-(al+bm+cn)(-al+bm+cn)(al-bm+cn)(al+bm-cn).$$

16. Prove that $\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & x & a & a \\ b & b & x & b \\ c & c & c & x \end{vmatrix} = (x-a)(x-b)(x-c).$

17. Prove that $\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & l+a & a_3 & a_4 \\ b & b & m+b & b_4 \\ c & c & c & n+c \end{vmatrix} = l.\ m.\ n.$

18. Prove that $\begin{vmatrix} 1 & 2x\cos a & x^2 \\ x^2 & 1 & 2x\cos a \\ 2x\cos a & x^2 & 1 \end{vmatrix} = 1+2x^3\cos 3a+x^6$.

19. Prove that $\begin{vmatrix} (al-pm)^2 & a^2+p^2 & (am+pl)^2 \\ (bl-qm)^2 & b^2+q^2 & (bm+ql)^2 \\ (cl-rm)^2 & c^2+r^2 & (cm+rl)^2 \end{vmatrix} = 0.$

20. If A_1, B_1, C_1, etc. are cofactors of a_1, b_1, c_1, ...in $[a_1\ b_2\ c_3]$ prove that

$$\begin{vmatrix} B_1+C_1 & C_1+A_1 & A_1+B_1 \\ B_2+C_2 & C_2+A_2 & A_2+B_2 \\ B_3+C_3 & C_3+A_3 & A_3+B_3 \end{vmatrix} = 2\,[a_1\ b_2\ c_3]^2.$$

21. Solve the simultaneous equations

$$x+y+z+w+1=0.$$
$$1x+2y+3z+4w=0.$$
$$1^2x+2^2y+3^2z+4^2w=0,$$
and $$1^3x+2^3y+3^3z+4^3w=0.$$

22. Solve the simultaneous equations

$$x+y+z+w=0,$$
$$ax+(a+1)y+(a+2)z+(a+3)w=0,$$
$$a^2x+(a+1)^2y+(a+2)^2z+(a+3)^2w=0,$$
$$a^3x+(a+1)^3y+(a+2)^3z+(a+3)^3w=6.$$

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* CHAPTER III

Analytical Trigonometry
COMPLEX NUMBERS

quantities (Loney II 7)

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3·01. Obviously no positive number can satisfy an equation of the form $x+1=0$. Similarly, if we admit only positive integers, the equation $2x-1=0$, is meaningless. Thus, in order that the equation $ax+b=0$ where a and b are integers, may be always solvable, it is necessary to recognise new types of numbers–the so-called rational or commensurable, and negative numbers. Again, it is seen without much difficulty that the equation $x^2-2=0$, cannot be satisfied by any rational number either positive or negative. In order to overcome this difficulty the so-called irrational numbers are introduced into mathematics. A further extention of the concept of number is found necessary: for, since the squares of real numbers are always positive, no number either positive or negative. rational or irrational, can satisfy an equation of the form $x^2+1=0$ Numbers which satisfy equations of this type are called complex or imaginary numbers. It is thus seen how in order to extend the range of application of mathematical results, it is necessary to generalise the idea of number. In this chapter an attempt is made to build up the algebra of complex numbers.

3·1. Definition of a Complex Number. An *ordered pair of real numbers* symbolically represented by (x, y) is called a *complex number*: thus $(-1, 2)$, $(1, 0)$, $(\sqrt{2}, \sqrt[3]{3})$, denote complex numbers. It should be noticed that the order in which the two real numbers are written in the symbol, is significant. Thus $(2, 3)$ and $(3, 2)$ are two distinct complex numbers. The first number x is called the *real part* and the second number y, the *imaginary part*, of the complex number (x, y).

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for 3,3',2 , 3,3-1,2,3

The student is advised to revise his knowledge of the Fundamental Laws of Algebra. 'A New Algebra' by Barnard and Child vols, I, II, is most suited for the purpose. It is one of the most thoughtfully written books on the elements of Algebra.

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3·2. Equality of Complex numbers. If the real parts and imaginary parts of two complex numbers are separately equal, then they are said to be equal. Thus, (1, 2) and (2−1, 1+1) are two equal complex numbers. This relation between them is shortly written thus

$$(1, 2) = (2-1, 1+1).$$

3·3. Addition of Complex Numbers. From any two complex numbers $z_1 \equiv (x_1, y_1)$ and $z_2 \equiv (x_2, y_2)$, we derive a third complex number (x_1+x_2, y_1+y_2) of which the real part is obtained by adding the real part of the complex number z_2 to the real part of z_1; the imaginary part is similarly derived. The complex number thus obtained is said to be the result of adding z_2 to z_1. The relation connecting them is expressed thus;

$$z_1+z_2 \equiv (x_1, y_1)+(x_2, y_2)=(x_1+x_2, y_1+y_2).$$

Denoting any three complex numbers by single letters z_1, z_2, z_3, their sum is defined by the equation

$$z_1+z_2+z_3=(z_1+z_2)+z_3.$$

In this way, we define the sum of four, five,..., complex numbers successively.

3·31. *Complex numbers obey the Commutative Law for Addition.*

Let (x_1, y_1) and (x_2, y_2) be any two complex numbers; then by definition (§ 3·3)

$$(x_1, y_1)+(x_2, y_2)=(x_1+x_2, y_1+y_2),$$

and

$$(x_2, y_2)+(x_1, y_1)=(x_2+x_1, y_2+y_1).$$

Now x_1, y_1, x_2, y_2 are all real;

$\therefore\ x_1+x_2=x_2+x_1$ and $y_1+y_2=y_2+y_1$;

$\therefore$ by § 3·2, $(x_1+x_2, y_1+y_2)=(x_2+x_1, y_2+y_1)$.

Thus if the two complex numbers are denoted by z_1, z_2 we have

$$z_1+z_2=z_2+z_1.$$

3·32. *Complex numbers obey the Associative Law for Addition.*

If z_1, z_2, z_3 denote any three complex numbers, we have to prove that

$$(z_1+z_2)+z_3=z_1+(z_2+z_3).$$

Let the complex numbers denoted by z_1, z_2, z_3 be $(x_1, y_1), (x_2, y_2), (x_3, y_3)$; then, by § 3·3,

$$z_1+z_2=(x_1, y_1)+(x_2, y_2)=(x_1+x_2, y_1+y_2);$$
$$\therefore\ (z_1+z_2)+z_3=(x_1+x_2\ \ y_1+y_2)+(x_3, y_3)$$
$$=[(x_1+x_2)+x_3,\ (y_1+y_2)+y_3].$$

Similarly $z_1+(z_2+z_3)=[x_1+(x_2+x_3),\ y_1+(y_2+y_3)]$.

Now x_1, y_1; x_2, y_2; x_3, y_3 are all real; and hence

$(x_1+x_2)+x_3=x_1+(x_2+x_3)$ and $(y_1+y_2)+y_3=y_1+(y_2+y_3)$;

$\therefore$ by § 3·2, $[(x_1+x_2)+x_3,\ (y_1+y_2)+y_3]$

$$=[x_1+(x_2+x_3),\ y_1+(y_2+y_3)].$$

Thus, $(z_1+z_2)+z_3=z_1+(z_2+z_3)$.

3·33. *Given two complex numbers z_1, z_2, it is always possible to determine uniquely a complex number z such that $z_2+z=z_1$.*

Assume the existence of such a number z and suppose z, z_1, z_2 denote (x, y), (x_1, y_1), (x_2, y_2), then

$$z_2+z=(x_2, y_2)+(x, y)=(x_2+x, y_2+y).$$

Now $z_2+z=z_1$, $\therefore\ (x_2+x, y_2+y)=(x_1, y_1)$.

Hence, by § 3·2, $x_2+x=x_1$ and $y_2+y=y_1$:

$$\therefore\ x=x_1-x_2 \text{ and } y=y_1-y_2.$$

Since all the above steps are reversible, we see that (x_1-x_2, y_1-y_2) is the required number z.

This uniquely determined complex number (x_1-x_2, y_1-y_2) is defined to be the result of subtracting the complex number (x_2, y_2) from (x_1, y_1). The relation connecting them is expressed in the form

$$(x_1, y_1)-(x_2, y_2)=(x_1-x_2, y_1-y_2).$$

Now it is easy to see that

$$(x_1, y_1)+(-x_2, -y_2)=(x_1-x_2, y_1-y_2).$$

Hence the effect of subtracting (x_2, y_2) is the same as that of adding $(-x_2, -y_2)$. Thus *the operation of subtraction does not essentially differ from that of addition.*

3·4. *Any complex number $z=(x, y)$ can be put into the form $(r\cos p, r\sin p)$ where r is a positive number.*

Assuming that the transformation is possible, we have

$$z=(x, y)=(r\cos p, r\sin p),$$
$$\therefore\ x=r\cos p \text{ and } y=r\sin p. \qquad (1)$$

Squaring and then adding these equations, we get

$$x^2+y^2=r^2(\cos^2 p+\sin^2 p)=r^2,\ \therefore\ r=\pm\sqrt{(x^2+y^2)}. \qquad (2)$$

Choosing the upper sign for the value of r and substituting this value in (1), we have

$$\cos p=x/\sqrt{(x^2+y^2)},\ \sin p=y/\sqrt{(x^2+y^2)}. \qquad (3)$$

Now, $\sqrt{(x^2+y^2)}$ is numerically not less than either of the quantities x and y; $\therefore$ there is a unique angle lying between $-\pi$ and π, whose cosine and sine are respectively $x/\sqrt{(x^2+y^2)}$, $y/\sqrt{(x^2+y^2)}$. Denoting this angle by t we have $\cos p=\cos t$: and $\sin p=\sin t$; $\therefore p=t+2n\pi$ where n is an integer positive or negative. r is called the *modulus* of z and is denoted by $|z|$. t is called the *principal value of the amplitude* or simply *the amplitude* of z. The forms $(r\cos t, r\sin t)$ of a complex number will be hereafter referred to as the *normal form*

When $x=-1, y=0$, $\cos t=-1$, $\sin t=0$ $\therefore t=\pi$, or $-\pi$: we agree to choose π in this case.

Example 1. Put $(1,0)$ into the normal form.

$|(1,0)|=\sqrt{(1^2+0)}=1$: $\therefore$ the amplitude is determined by $\cos t=1$ and $\sin t=0$ and so the value of t that lies between $-\pi$ and π is 0. Hence

$$(+1,0)=(\cos 0,\ \sin 0).$$

Example 2 Put $(-1, 0)$ into the normal form.

Here $|(0,1)|=1$, and the amplitude is determined by equations $\cos t=-1$, $\sin t=0$: $\therefore t=\pi$; $\therefore$ $(-1;\ 0)=(\cos\pi,\ \sin\pi)$.

Example 3. Put $(0,1)$ into the normal form.

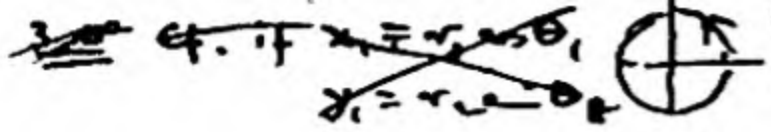
Here $|(0,1)|=1$. $\therefore$ $\cos t=0$, $\sin t=1$; $\therefore$ the angle which satisfies these and lies between $-\pi$ and π is $\pi/2$.

$$\therefore\ (0,1)=[\cos(\pi/2),\ \sin(\pi/2)].$$

Example 4. Put $(1,\ \sqrt{3})$ into the normal form.

Here $|(1,\ \sqrt{3})|=\sqrt{(1+3)}=2$. $\therefore$ the amplitude is determined by the equations $\cos t=1/2$ and $\sin t=\sqrt{3}/2$; $\therefore$ $t=\pi/3$;

$$\therefore\ (1,\ \sqrt{3})=(2\cos\pi/3,\ 2\sin\pi/3,).$$



Examples. III. a.

Put each of the following into the normal form:

1. (1, 1). 2. (1,−1). 3. (−1, 1). 4. (−1,−1).
5. (1,−√3). 6. (−1, √3). 7. (0, −1).
8. (−1,−√3). 9. (√3, 1). 10. (−√3,−1).
11. (−√3, 1). 12. (√3,−1). 13. (3, √3).

3·5. Multiplication of Complex Numbers.

The result of multiplying $z \equiv (x_1, y_1,)$ by $z_2 \equiv (x_2, y_2)$ is defined to be the complex number $(x_1x_2 - y_1y_2,\ x_1y_2 + y_1x_2)$. We express the relation thus,

$$z_1 \times z_2 \equiv (x_1, y_1) \times (x_2, y_2) = (x_1x_2 - y_1y_2,\ x_1y_2 + y_1x_2).$$

The product of three complex numbers z_1, z_2, z_3 is defined by

$$z_1 \times z_2 \times z_3 = (z_1 \times z_2) \times z_3.$$

In this way we can successively define the product of three, four, five, ...complex numbers. The above form of the product of two complex numbers is rather complicated. It however, simplifies considerably if the normal form of complex numbers is used. Thus if

$$z_1 = (r_1\cos p_1,\ r_1\sin p_1),\quad z_2 = (r_2\cos p_2,\ r_2\sin p_2)$$

$z_1\ z_2$ is equal to

$$(r_1r_2\cos p_1\cos p_2 - r_1r_2\sin p_1\sin p_2,\ r_1r_2\cos p_1\sin p_2 + r_1r_2\sin p_1\cos p_2)$$

or $[\ r_1r_2(\cos p_1\cos p_2 - \sin p_1\sin p_2),\ r_1r_2(\cos p_1\sin p_2 + \sin p_1\cos p_2)\ ]$

Now,
$$\cos p_1\cos p_2 - \sin p_1\sin p_2 = \cos(p_1 + p_2),$$
$$\cos p_1\sin p_2 + \sin p_1\cos p_2 = \sin(p_1 + p_2).$$
$$\therefore\ z_1.\ z_2 = [r_1r_2\cos(p_1 + p_2),\ r_1r_2\sin(p_1 + p_2)\ ].$$

3·51. *Complex numbers obey the Commutative Law for Multiplication.*

For, in the notation of §3·5:

$$z_1\ z_2 = [\ r_1\ r_2\cos(p_1 + p_2),\ r_1\ r_2\sin(p_1 + p_2)\ ]$$

and
$$z_2\ z_1 = [\ r_2\ r_1\cos(p_2 + p_1),\ r_2\ r_1\sin(p_2 + p_1)\ ].$$

Now r_1, r_2, p_1, p_2, are real;

$$\therefore\quad r_1\ r_2 = r_2\ r_1 \text{ and } p_1 + p_2 = p_2 + p_1;$$

and hence
$$z_1\ z_2 = z_2\ z_1.$$

3·52. *Complex numbers obey the Associative Law for Multiplication.*

For, in the notation of § 3·5,

$$z_1 z_2 = [r_1 r_2 \cos(p_1+p_2),\ r_1 r_2 \sin(p_1+p_2)];$$

$$\therefore\ (z_1 z_2).z_3 = [r_1 r_2 \cos(p_1+p_2), r_1 r_2 \sin(p_1+p_2)] \times (r_3 \cos p_3,\ r_3 \sin p_3)$$

$$= [(r_1 r_2) r_3 \cos(p_1+p_2+p_3),\ (r_1 r_2) r_3 \sin(p_1+p_2+p_3)].$$

Similarly $z_1.(z_2, z_3)$

$$= [r_1(r_2 r_3) \cos(p_1+p_2+p_3),\ r_1(r_2 r_3) \sin(p_1+p_2+p_3)].$$

Since $r_1, r_2, r_3, p_1, p_2 \ldots$ are real, we have

$$(r_1 r_2) r_3 = r_1(r_2 r_3) \text{ and } (p_1+p_2)+p_3 = p_1+(p_2+p_3);$$

$$\therefore (z_1 z_2) z_3 = z_1 (z_2 z_3).$$

3·53. *Complex numbers obey the Distributive Law of Multiplication*

If z_1, z_2, z_3 are any three complex numbers, we have to prove that

$$z_1(z_2+z_3) = z_1 z_2 + z_1 z_3.$$

Let z_1, z_2, z_3 stand for (x_r, y_r) where $r = 1, 2, 3$; then,

$$z_2 + z_3 = (x_2, y_2) + (x_3, y_3) = [(x_2+x_3), (y_2+y_3)]$$

$$\therefore\ z_1(z_2+z_3) = (x_1, y_1)(x_2+x_3,\ y_2+y_3)$$

$$= [x_1(x_2+x_3) - y_1(y_2+y_3), x_1(y_2+y_3) + y_1(x_2+x_3)] \qquad (\S\ 3·5)$$

$$= [(x_1x_2 - y_1y_2) + (x_1x_3 - y_1y_3),\ (x_1y_2 + y_1x_2) + (x_1y_3 + y_1x_3)]. \quad (1)$$

Now $z_1 z_2 = (x_1\ y_1)(x_2, y_2) = (x_1x_2 - y_1y_2 \quad x_1y_2 + y_1x_2)$

and $z_1 z_3 = (x_1, y_1)(x_3\ y_3) = (x_1x_3 - y_1y_3,\ x_1y_3 + y_1x_3)$.

Hence $z_1 z_2 + z_1 z_3$

$$= [(x_1x_2 - y_1y_2) + (x_1x_3 - y_1y_3), (x_1y_2 + y_1x_2) + (x_1y_3 + y_1x_3)]; \quad (2)$$

and so from (1), and (2) we have

$$z_1(z_2+z_3) = z_1z_2 + z_1z_3.$$

3·531. The real number 0, possesses properties symbolised by the equations

(1) $a \pm 0 = a$,

(2) $a \times 0 = 0 \times a = 0$,

(3) Further, if $a \times b = 0$ then at least one of the factors a, b must be 0.

The complex number (0, 0) is characterised by similar properties; for,

(1′) $(x, y) \pm (0, 0) = (x \pm 0,\ y \pm 0) = (x, y)$;

(2′) $(x, y), (0, 0) = [(x, 0-y.0),\ (x.0+y.0)] = (0, 0)$.

(3′) Again, if the product $(r_1 \cos p_1, r_1 \sin p_1), \times (r_2 \cos p_2, r_2 \sin p_2)$

is equal to (0, 0), then one of them, at least, must reduce to (0, 0). For

$(r_1\cos p_1, r_1\sin p_1)(r_2\cos p_2, r_2\sin p_2) = [r_1r_2\cos(p_1+p_2), r_1r_2\sin(p_1+p_2)]$

and $[r_1r_2\cos(p_1+p_2),\ r_1r_2\sin(p_1+p_2)\ ] = (0, 0)$;

and hence $r_1r_2\cos(p_1+p_2)=0,\ \ r_1r_2\sin(p_1+p_2)=0.$

Now $\cos(p_1+p_2)$ and $\sin(p_1+p_2)$, cannot be simultaneously zero :

$\therefore r_1r_2=0$; $\therefore$ at least one of the numbers r_1, r_2 must be 0. Hence at least one of the given complex numbers must reduce to (0, 0).

For these reasons we shall hereafter denote the complex number (0, 0) by the same symbol 0.

3·54. *Given two complex numbers z_1 and z_2, it is possible to determine uniquely a complex number z such that $z.\,z_2=z_1$, provided $z_2\neq 0$*

Assume that z exists and let z, z_1, z_2 represent $(r\cos p, r\sin p)$ $(r_1\cos p_1, r_1\sin p_1)$, $(r_2\cos p_2, r_2\sin p_2)$ respectively; so that

$$(r\cos p, r\sin p)(r_2\cos p_2, r_2\sin p_2) = (r_1\cos p_1, r_1\sin p_1)$$

or $$[rr_2\cos(p+p_2),\ rr_2\sin(p+p_2)\ ] = (r_1\cos p_1, r_1\sin p_1);$$

$$\therefore rr_2=r_1. \quad (1) \quad \text{and} \quad p+p_2=2n\pi+p_1. \quad (2)$$

where n is an integer. From (1), if $r_2\neq 0$ *i.e.* if $z_2\neq 0$, (§3·531) we get $r=r_1/r_2$; and from (2), $p=2n\pi+p_1-p_2$, and hence

$$z = \left(\frac{r_1}{r_2}\cos(p_1-p_2),\ \frac{r_1}{r_2}\sin(p_1-p_2)\right) \qquad \text{if } z_2\neq 0.$$

This number z is defined to be the result of dividing z_1 by z_2 and the relation is expressed as usual by

$$z=z_1/z_2=z_1\div z_2.$$

Now $$\left(\frac{r_1}{r_2}\cos\overline{p_1-p_2},\ \frac{r_1}{r_2}\sin\overline{p_1-p_2}\right)$$

$$=(r_1\cos p_1, r_1\sin p_1)\left(\frac{1}{r_2}\cos(-p_2),\ \frac{1}{r_2}\sin(-p_2)\right).$$

Hence the effect of dividing z_1 by z_2 is the same as that of multiplying z_1 by $\left(\frac{1}{r_1}\cos(-p_2),\ \frac{1}{r_1}\sin(-p_2)\right)$ It is thus seen that *the operation of division is not essentially different from that of multiplication.*

3·6. Complex numbers like $(x, 0)$, $(y, 0)$, $(z, 0)$ whose ima-

ginary parts are each equal to zero, behave like the corresponding real numbers x, y,.....This is seen from the following equations:

$$(x, 0)+(y, 0)=(x+y, 0),$$
$$(x, 0)-(y, 0)=(x-y, 0),$$
$$(x, 0)\times(y, 0)=(xy, 0),$$

and
$$(x, 0)\div(y, 0)=(x\div y, 0).$$

For these reasons, we agree to denote them by the simpler symbols x, y, z,..... .

3·61. Complex number (0, 1).

By § 3·5, $(0, 1)\times(0, 1)=(-1, 0)=-1$, (§ 3·6).

If we define powers of a complex number as in the algebra of real numbers, it follows that (0, 1) is a root of the equation $z^2+1=0$. It is usual to denote this number by i. We see easily that the normal form of i is

$$(\cos\tfrac{1}{2}\pi, \sin\tfrac{1}{2}\pi).$$

3·62. It is easy to prove that $(0, y)=(0, 1)\times(y, 0)$. Now, by § 3·61, $(0, 1)=i$, and by §3.6, $(y, 0)=y$; $\therefore (0, y)=iy$. Again $(x, y)=(x, 0)+(0, y)$. Now by § 3·05 $(x, 0)=x$; and we have just shown that $(0, y)=iy$. Therefore

$$(x, y)=x+iy.$$

3·63. So, the whole of the foregoing theory reduces to this:—

The complex number (x, y) is the same as $x+iy$ where i is subject to the same fundamental laws of algebra as real numbers, with this additional property that $i\times i=i^2=-1$; further if $x+iy=x'+iy'$, then $x=x'$ and $y=y'$. e. g.

$$(x+iy)^2=x^2+2ixy+i^2y^2=x^2-y^2+i2xy;$$
$$(a+ib)(c+il)=ac+aid+ibc+i^2bd=ac+iad+ibc-bd$$
$$=ac-bd+i(ad+bc).$$

Examples. III. b.

Establish the following identities referring to the law applied at every stage of the transformation; the symbols represent complex numbers:

1. $(a+b)(c+d)=ac+bc+ad+bd$.
2. $(a^2-b^2)=(a+b)(a-b)$.
3. $(x+a)(x+b)(x+c)=x^3+(a+b+c)x^2+(ab+bc+ca)x+abc$.

4. If $(z-a)(z-b)=0$, then z must be equal to one of the quantities a, b at least.

5. If $az^2+2hz+c=0$, prove that $z=[-h\pm\sqrt{(h^2-ac)}]/a$.

3·7. Argand's (1806) Diagram and the Geometrical Representation of Complex Numbers.

Take any two rectangular axes $X'OX$ and $Y'OY$ as in coordinate geometry, and plot the pt. (x, y). Call this points P. Then the

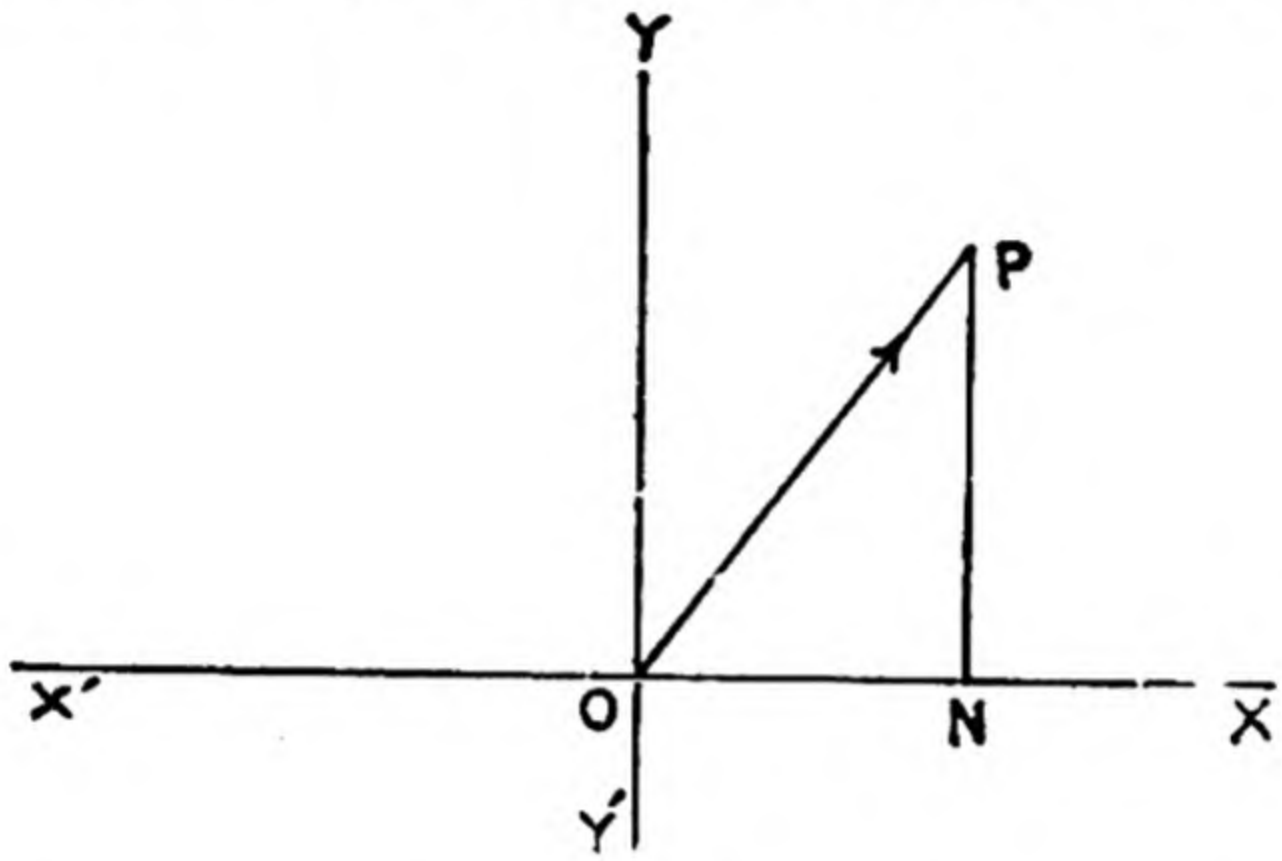


Fig. 1.

directed line or vector $\overline{OP}$ or the point P is said to represent the complex number $z\equiv x+iy$. It is thus seen that, corresponding to every complex number, there is a definite point in the plane XOY and vice versa. This figure is known as *Argand's Diagram*. If NP is drawn perpendicular to $X'OX$ it is easy to see from the right-angled triangle ONP that

$$ON=x,\ NP=y,\ \therefore OP=\sqrt{(x^2+y^2)}=|z|. \qquad (\S 3\cdot 4.)$$

Again $\angle XOP$ is such that its cosine and sine are respectively equal to $x/\sqrt{(x^2+y^2)}$, $y/\sqrt{(x^2+y^2)}$ and hence it is equal to t or differs from it by a multiple of 2π, where t is the amplitude of z, (§ 3·4).

3·71. *The Geometrical interpretation of the sum* $z_1+z_2+z_3$ *where* z_1, z_2, z_3 *are any complex numbers.* Let P_1, P_2, P_3 represent z_1, z_2, z_3, in the Argand's diagram. Draw P_1Q equal and parallel to, and in the same sense as, OP_2. Draw QR equal and parallel to, and in the same sense as, OP_3. Then the point R or the vector $\overline{OR}$ represents the sum $z_1+z_2+z_3$.

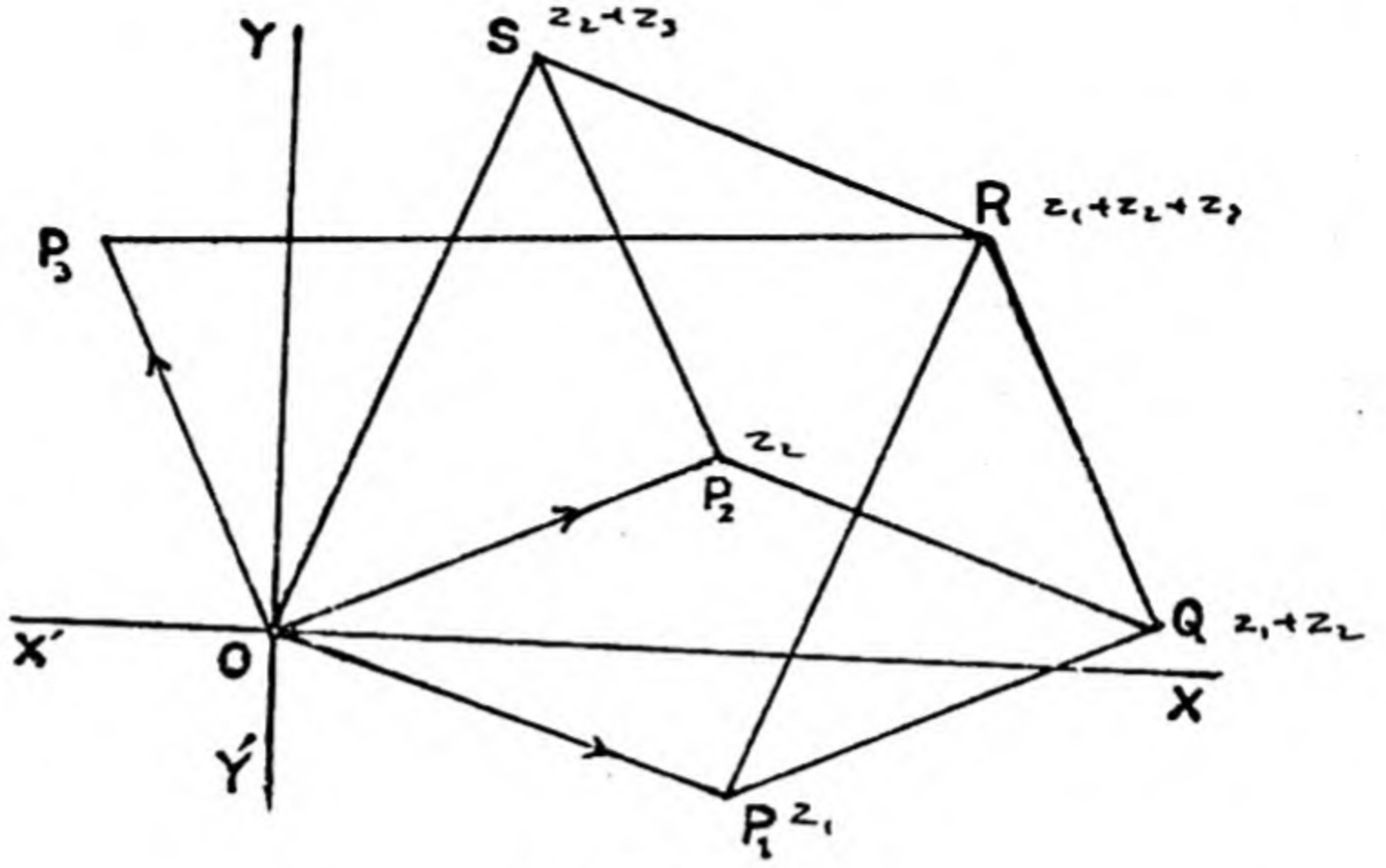


Fig. 2.

Let z_1, z_2, z_3 stand for x_1+iy_1; x_2+iy_2, x_3+iy_3, so that the coordinates of P_1, P_2, P_3 are (x_1, y_1), (x_2, y_2), (x_3, y_3). Now the abscissa of R is equal to the projection of OR on OX. ∴ it is equal to the sum of projections of OP_1, P_1Q, QR *i. e.* of OP_1, OP_2, OP_3, on OX. Now the projections of OP_1, OP_2, OP_3, are respectively x_1, x_2, x_3.

Hence the abscissa of $R = x_1+x_2+x_3$.

Similarly the ordinate of $R = y_1+y_2+y_3$.

and so the point R represents the complex number

$$\overline{x_1+x_2+x_3} + i\,\overline{y_1+y_2+y_3}.$$

This complex number is clearly equal to

$$(x_1+iy_1)+(x_2+iy_2)+(x_3+iy_3) \text{ or } z_1+z_2+z_3.$$

In the same way it is possible to construct the sum of any number of complex numbers; and in view of remarks made[61] at the end of § 3·33, it is seen that it is possible to construct geometrically an expression like

$$z_1+z_2-z_3-z_4+z_5\ .$$

3·711. In fig. 2, since P_1Q is equal and parallel to OP_2, $OP_1\,QP_2$ is a parallelogram. ∴ P_2Q is equal and parallel to and is drawn in the same sense as OP_1. Thus the point O is reached by drawing P_2Q equal and parallel to and in the same sense as OP_1. Thus, we have *the geometrical verification of the commutative Law for Addition*

3·712. In the same fig. 2, if the parallelogram P_2QRS is completed, then P_2S is equal and parallel to and is drawn in the same sense as QR, and QR is equal and parallel to and is drawn in the same sense as OP_3. Hence S represents z_2+z_3. Further it is easily proved that OP_1RS is a parallelogram. Hence R represents $z_1+z_2+z_3$ also. Thus we have the *geometrical verification of the Associative Law for Addition.*

3·713. In the same figure 2. it is evident from geometrical considerations, that the sum of lengths OP_2, P_2Q, QR is not less than the length OR. Now, OP_2, P_2Q, QR, OR are respectively equal to OP_2, OP_1, OP_3 and OR which represent the moduli of z_2, z_1, z_3 and $z_1+z_2+z_3$ Thus we have

$$|z_1|+|z_2+z_3| \leqslant |z_1+z_2+z_3| ;$$

or
$$\sqrt{(x_1^2+y_1^2)}+\sqrt{(x_2^2+y_2^2)}+\sqrt{(x_3^2+y_3^2)} \leqslant \sqrt{[(x_1+x_2+x_3)^2+(y_1+y_2+y_3)^2]} .$$

This theorem can be obviously generalised. It is a result of great importance. Since $x=x+io$ where x is real, we have

$$|x|=|x+io|=+\sqrt{(x^2)}=\text{the absolute value of } x;$$

We thus get the important particular case:–

If $x, y, z, \ldots$ are any real numbers then the sum of their absolute values is never less than the absolute value of their sum

e. g. None of the quantities $|-2+3|$, $|-2-3|$, $|2-3|$, is greater than

$$|\pm 2|+|\pm 3| .$$

Example 1, If z_1, z_2 represent points, A and B in the Argand's diagram, prove that M, the middle point of AB is represented by $(z_1+z_2)/2$.

By §3·71,
$$\overline{OB}=\overline{OA}+\overline{AB}, \therefore \overline{AB}=\overline{OB}-\overline{OA}=z_2-z_1,$$
$$\therefore AM=(z_2-z_1)/2.$$
$$\therefore \text{ by §3·71, } \overline{OM}=\overline{OA}+\overline{AM}=z_1+(z_2-z_1)/2.=(z_1+z_2)/2.$$

Example 2. Prove that lines connecting middle points of opposite pairs of sides of a quadrangle bisect one another.

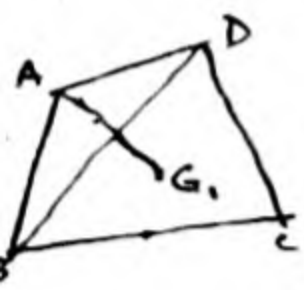
Let A, B, C, D be the vertices and let them represent z_1, z_2, z_3, z_4 in the Argand's diagram. Let L, M be the middle points of AB, CD; then by *Example* 1, above, L and M represent $(z_1+z_2)/2$ and $(z_3+z_4)/2$. Therefore the middle point of LM represents

$$[(z_1+z_2)/2+(z_3+z_4)/2]/2 \text{ or } (z_1+z_2+z_3+z_4)/4.$$

The symmetry of this result shows that the middle point of the line joining the middle points of BC, DB, and the middle point of the line joining the middle points of AC, BD must each of them be represented by $(z_1+z_2+z_3+z_4)/4$. The required result is now quite obvious,

Examples. III. c.

1. If z_1, z_2, z_3 represent vertices of a Δ in the Argand's diagram, prove that $(z_1+z_2+z_3)/3$ represents its centroid.



2. In a quadrangle $ABCD$, lines joining A, B, C, D, to the centroids of ΔBCD, ΔCDA, ΔDAB, ΔABC, meet in a point and divide one another in the ratio, 3:1.

3. If z_1, z_2 represent B, C then the point D which divides BC in the ratio $m:n$ is given by $(nz_1+mz_2)/(m+n)$.

4. If $lz_1+mz_2+nz_3=0$, where l, m, n, are real numbers such that $l+m+n=0$, then points representing z_1, z_2, z_3 in the Argand's diagram are collinear.

5. Prove that $z+z_1$, $z+z_2$, $z+z_3$, $z+z_4$ where z_1, z_2, z_3, z_4 are complex numbers with the same modulus, represent concyclic points in the Argand's diagram.

6. The figure whose vertices represent z_1, z_2, z_3, z_4...is equal, in all respects, to the figure whose vertices represent $z+z_1$, $z+z_2$, $z+z_3$,.....in the Argand's diagram.

7. The sides of a ΔABC are divided in the same ratio at L, M, N. Prove that the centroids of triangles ABC, LMN coincide.

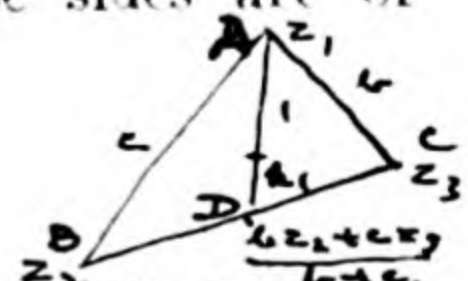
8. The origin of the Argand's diagram is transferred to a pt. representing the complex number a, the directions of axes remaining the same. If z, Z, represent the same pt. in the old and new Argand's diagrams, prove that

$$z=Z+a.$$

9. If z_1, z_2, z_3 are the vertices of a Δ, whose sides are of lengths a, b, c, prove that

$$(az_1+bz_2+cz_3)\div(a+b+c)$$

represents its incentre.



3·72. *Geometrical interpretation of the multiplication of complex numbers.*

Let P_1, P_2 represent $r_1\cos p_1 + ir_1\sin p_1$, and $r_2\cos p_2 + ir_2\sin p_2$ in the Argand's diagram ; so that OP_1, OP_2 are respectively equal to r_1, r_2 and $\angle\ XOP_1$, $\angle\ XOP_2$, equal to p_1 and p_2. Cut off a unit distance OU along OX. Join UP_1. Construct $\triangle OP_2R$ similar

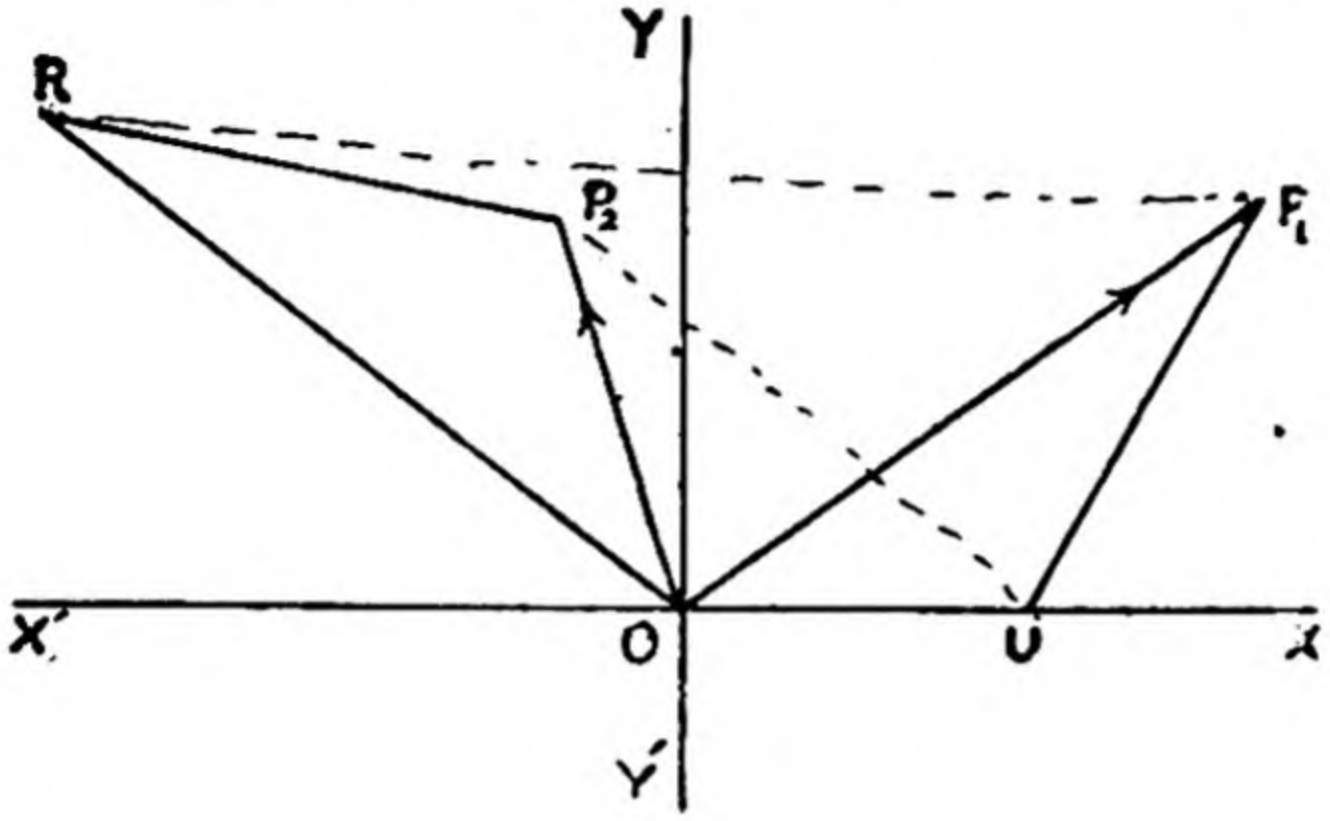


Fig. 3.

to and described in the same sense as, $\triangle OUP_1$. Then the point R or the vector $\overline{OR}$ represents the product. For, since $\triangle OUP_1$ and $\triangle OP_2R$ are similar:

$$OR/OP_2 = OP_1/OU;\ \therefore\ OR = OP_1.OP_2 = r_1 r_2,$$

Again $$\angle XOR = \angle XOP_2 + \angle P_2OR = p_1 + p_2.$$

Hence, by §3·7, R represents

$$r_1r_2\cos(p_1+p_2) + ir_1r_2\sin(p_1+p_2)$$

or $$[\,r_1\cos p_1 + ir_1\sin p_1] \times [\,r_2\cos p_2 + ir_2\sin p_2.]. \qquad (\S\ 3{\cdot}5)$$

The point R can also be constructed in the following manner: Cut off OL along OP_1, so that $OL = r_2\ OP_1$ and then rotate OL through p_2, the sense of rotation being determined by the sign of p_2. Thus we see that "*The multiplication of two complex numbers consists of a stretching and a turning.*" (*B. U. 1924 Hons. II*).

3·721. In the same Fig. 3. join UP_2 and P_1R; then, it is easy to show that $\triangle OP_1R$ is similar to and is described in the same sense, as $\triangle OUP_2$. We thus verify the Commutative Law of Multiplication.

3·722. *Geometrical verification of the distributive law.*

Let P_1, P_2, P_3, represent any three complex numbers,

$z_1 = r_1\cos p_1 + ir_1\sin p_1$, $z_2 \equiv r_2\cos p_2 + ir_2\sin p_2$ $z_3 = r_3\cos p_3 + ir_3\sin p_3$,

in the Argand's diagram. Cut off a unit distance OU along OX.

Draw $P_1 Q$ equal and parallel to, and in the same sense as, OP_2. Join UP_1, UQ. Describe $\triangle OP_3 S$ and $\triangle OP_3 R$ similar to and in the same sense as, $\triangle OUP_1$ and $\triangle OUQ$ respectively. Join SR.

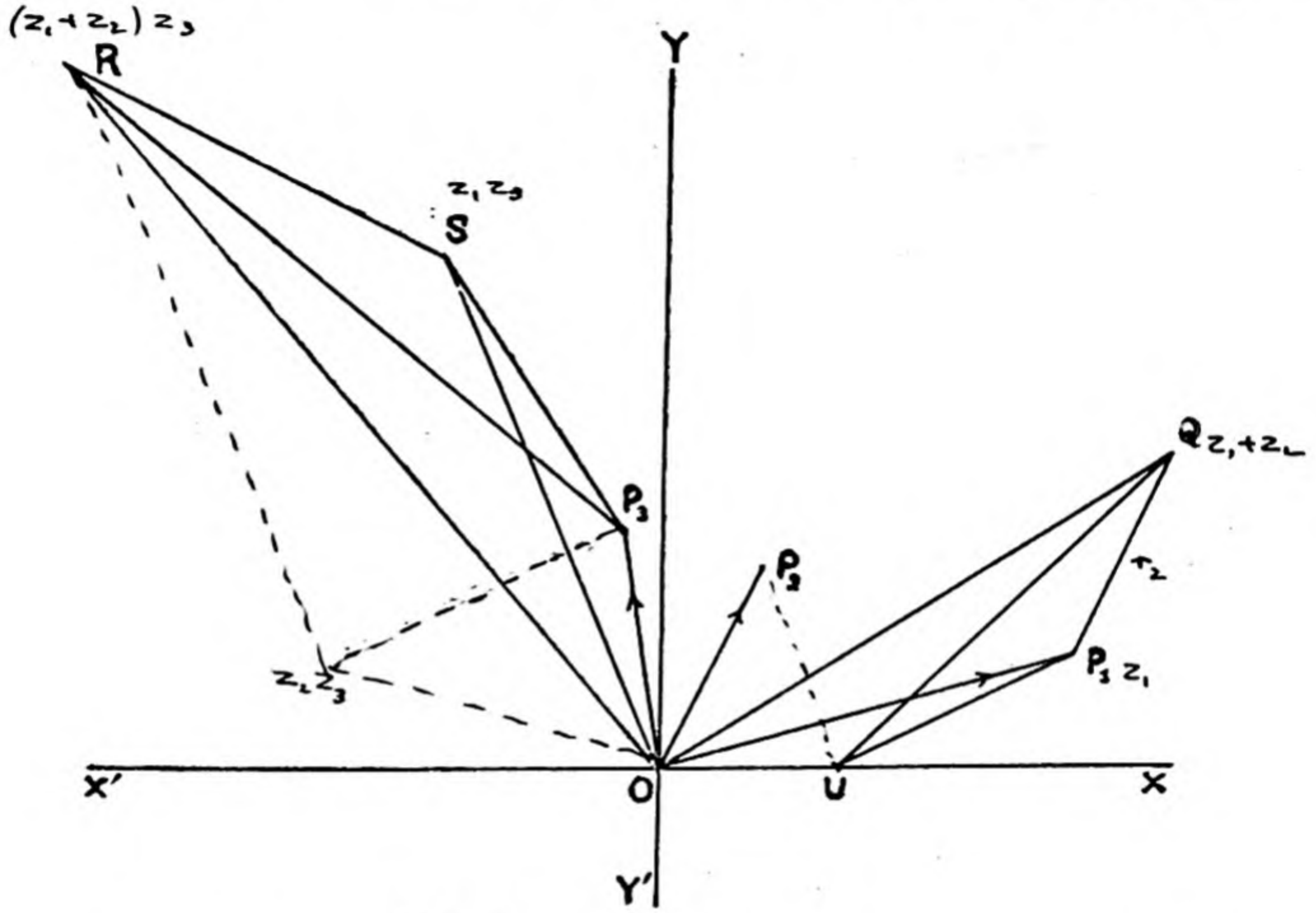


Fig. 4.

Now by § 3·71, Q represents $z_1 + z_2$ and therefore, by §3·72, R represents $(z_1 + z_2)z_3$. Again by construction, $\triangle OP_3 S$ is similar to and is described in the same sense as, OUP_1. Hence, by §3·72, S represents $z_1 z_3$. Further, it is easy to prove that $\triangle OP_1 Q$ and $\triangle OSR$ are similar; therefore $SR/P_1Q = OS/OP_1$; but from similar $\triangle OUP_1$ and $\triangle OP_3 S$. we have

$$OS/OP_1 = OP_3/OU,$$

$$\therefore\ SR/P_1Q = OP_3/OU \text{ or } SR = OP_3 . P_1Q = r_2 r_3.$$

From the geometry of the figure it is easy to see that SR makes angle $p_2 + p_3$ with the positive direction of OX. Hence SR is equal and parallel to and is drawn in the same sense as, the vector representing $z_2 z_3$. Therefore by § 3·71, OR or the point R represents $z_1 z_3 + z_2 z_3$ Thus we have the geometrical verification of the theorem

$$(z_1 + z_2)z_3 = z_1 z_3 + z_2 z_3.$$

Example 1. If P, Q, R, S represent z_1, z_2, z_3, z_4 in the Argand's diagram and if $z_1z_4=z_2z_3$, prove that $\triangle POQ$ is similar to $\triangle ROS$ and that $\triangle POR$ is similar to $\triangle QOS$.

Let z_1, z_2, z_3, z_4 stand for $r_1\cos p_1+ir_1\sin p_1$, etc. then the condition $z_1z_4=z_2z_3$ reduces to

$r_1r_4[\cos(p_1+p_4)+i\sin(p_1+p_4)]=r_2r_3[\cos(p_2+p_3)+i\sin(p_2+p_3)]$,

$\therefore\ r_1r_4=r_2r_3$ and $p_1+p_4=p_2+p_3+2n\pi$, n being an integer.

$\therefore\ r_1/r_2=r_3/r_4$, and $p_2-p_1=p_4-p_3-2n\pi$.

Hence in the two $\triangle$s POQ and ROS, $\angle POQ=\angle ROS$, and the sides about these angles are proportional; $\therefore$ the triangles POQ, ROS are similar. The remaining part follows from the symmetry of the given condition.

Example 2. If L, M represent z_1, z_2 where $z_1^2+z_2^2=0$, prove that $\triangle LOM$ is isosceles and right-angled at O.

Let z_1 and z_2 stand for $r_1\cos p_1+ir_1\sin p_1$ and $r_2\cos p_2+ir_2\sin p_2$.

Now $z_1^2+z_2^2\equiv(z_1+iz_2)(z_1-iz_2)=0$; $\therefore z_1=iz_2$ or $-iz_2$.

i.e. $r_1\cos p_1+ir_1\sin p_1=(\cos\frac{1}{2}\pi\mp i\sin\frac{1}{2}\pi)(r_2\cos p_2+ir_2\sin p_2)$, (§3·61.)

$\therefore\ r_1(\cos p_1+i\sin p_1)=r_2[\cos(\frac{1}{2}\pi\mp p_2)+i\sin(\frac{1}{2}\pi\mp p_2)]$

$\therefore\ r_1=r_2$, and $p_1=\frac{1}{2}\pi\mp p_2+2n\pi$

where n is an integer; from these, the required result is easily deduced.

Examples. III. d.

1. Interpret geometrically the division of two complex numbers.

2. Verify geometrically the Associative Law for Multiplication of complex numbers

3. $\triangle OPQ$ and $\triangle ORS$ are similar and are described in the same sense. If L, and M divide PR, QS in the same ratio, prove that $\triangle OLM$ is similar to $\triangle OPQ$ or $\triangle ORS$.

[If P, Q, R, S represent z_1, z_2, z_3, z_4, then we have $z_1/z_2=z_3/z_4$. Now each of these equal ratios $=(lz_1+mz_3)/(lz_2+mz_4)$, where l and m are any real numbers.]

4. C, D, E, F represent z_1, z_2, z_3, z_4 in the Argand's diagram. If $CD\perp EF$, prove that $(z_1-z_2)^2+K(z_3-z_4)^2=0$ where K is any positive real number.

5. In the notation of *Example* 4, prove that the quad. $PQRS$ is a parallelogram if

$$[(z_1-z_2)^2-(z_3-z_4)^2][(z_1-z_3)^2-(z_2-z_4)^2][(z_1-z_4)^2-(z_2-z_3)^2]=0.$$

6. $\triangle ABC$ and $\triangle PQR$ are similar, where A, B, C, P, Q, R represent complex numbers z_1, z_2, z_3; zz_1, zz_2, zz_3;

$$\begin{vmatrix} zz_1 & zz_2 & zz_3 \\ z_1 & z_2 & z_3 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

7. $\triangle ABC$ whose vertices represent z_1, z_2, z_3, is right-angled and isosceles; prove that

$$[(z_1-z_2)^2+(z_1-z_3)^2][(z_2-z_1)^2+(z_2-z_3)^2][(z_3-z_1)^2+(z_3-z_2)^2]=0.$$

3·8. *The modulus of the product of any number of complex numbers is equal to the product of their moduli.*

The amplitude of the product of any number of complex numbers differs from the sum of their amplitudes by a multiple of 2π.

Let the complex numbers be $z_1=r_1(\cos p_1+i\sin p_1)$,

$$z_2=r_2(\cos p_2+i\sin p_2),\ldots,z_n=r_n(\cos p_n+i\sin p_n).$$

Now $z_1z_2=r_1r_2(\cos p_1+i\sin p_1)(\cos p_2+i\sin p_2)$

$$=r_1r_2[(\cos p_1\cos p_2-\sin p_1\sin p_2)\times i(\sin p_1\cos p_2+\cos p_1\sin p_2)]$$

$$=r_1r_2[\cos(p_1+p_2)+i\sin(p_1+p_2)].$$

$$z_1z_2z_3=(z_1z_2)z_3=r_1r_2r_3[\cos(p_1+p_2)+i\sin(p_1+p_2)](\cos p_3+i\sin p_3)$$

$$=r_1r_2r_3[\cos(p_1+p_2+p_3)+i\sin(p_1+p_2+p_3)]$$

Similarly

$$z_1z_2z_3z_4=r_1r_2r_3r_4[\cos(p_1+p_2+p_3+p_4)+i\sin(p_1+p_2+p_3+p_4)]$$

Repeating this process, we have

$$z_1z_2\ldots z_n=r_1r_2\ldots r_n[\cos(p_1+p_2+\ldots+p_n)+i\sin(p_1+p_2+\ldots+p_n)].$$

This result shows that the modulus of $z_1z_2z_3\ldots z_n$ is $r_1r_2\ldots\ldots r_n$ which is the product of the moduli of $z_1, z_2,\ldots z_n$. and that its amplitude is equal to $p_1+p_2+\ldots+p_n$ to a multiple of 2π. By putting $r_1=r_2=\ldots=r_n=1$ and $p_1=p_2=\ldots=p_n=p$, we get the important particular case

$$(\cos p+i\sin p)^n=\cos np+i\sin np.$$

3·9. Two complex numbers which differ only in the sign of their imaginary parts are said to be *conjugate complex numbers.* Thus $1+2i$, $1-2i$ are a pair of conjugate complex numbers. Points representing such a pair of numbers in the Argand's diagram are clearly images of each other in the axis of x. Following facts are

easily deduced from the definition of a pair of conjugate complex numbers :–

1. *Their moduli are equal; (2) their amplitudes are equal in magnitude but opposite in sign; (3) their sum and their product are real; (4) conversely, if two numbers are such that their sum and product are both real, then they are either real or they are a pair of conjugate complex numbers.*

3·91. *A rational function $f(z)$ where $z \equiv x+iy$ can be reduced to the from $X+iY$ where X and Y are rational functions of x and y with real coefficients.*

By the Binomial Theorem, we have

$$z^n=(x+iy)^n=x^n+{}_nC_1x^{n-1}yi+{}_nC_2x^{n-2}(yi)^2+{}_nC_3x^{n-3}(yi)^3+\ldots$$

Observing that $i^2=-1$, $i^3=-i$, $i^4=1$, $i^5=i\ldots$, we get

$$z^n=x^n-{}_nC_2x^{n-2}y^2+{}_nC_4x^{n-4}y^4-\ldots\ldots+i({}_nC_1x^{n-1}y-{}_nC_3x^{n-3}y^3+\ldots\ldots).$$

which is of the form $A+iB$ where A and B are polynomials in x, y with real coefficients. If $a_n=c_n+id_n$, then it is obvious that a_nz^n reduces to the same form. Again a polynomial in z consists of the sum of terms like $a_n z^n$; hence it also reduces to the form $P+iQ$. Finally since $f(z)$ is the quotient of two polynomials, and hence it can be put into the form $(P+iQ)/(R+iS)$.

Now
$$\frac{P+iQ}{R+iS}=\frac{(P+iQ)(R-iS)}{(R+iS)(R-iS)}=\frac{(PR+QS)+i(QR-PS)}{R^2+S^2}$$
$$=\frac{PR+QS}{R^2+S^2}+i\,\frac{QR-PS}{R^2+S^2}=X+iY$$

where X and Y are rational functions of X and Y with real coefficients.

3·92. *If $f(x+iy)\equiv A+iB$ where $f(z)$ is a rational function of z with real coefficients, then $f(x-iy)=A-iB$, A and B being rational functions of x, and y with real coefficients.*

Since $i^{2r}=(-i)^{2r}$, the terms which do not contain i are identical in $(x+iy)^n$ and $(x-iy)^n$. Again $\because (i)^{2r-1}=-(-i)^{2r-1}$, it follows that terms which involve i are identical but for sign, in the expansions of $(x+iy)^n$ and $(x-iy)^n$. Hence if $(x+iy)^n=X+iY$, then

$$(x-iy)^n=X-iY.$$

It follows that the theorem holds in the case of polynomials

in z with real coefficients. Thus if $f(x+iy)=(P+iQ)\div(R+iS')$, then $f(x-iy)=(P-iQ)\div(R-iS)$.

Now $(P+iQ)/(R+iS)=(P+iQ)(R-iS)\div(R^2+S^2)$

$$=[(PR+QS)+i(QR-PS)]\div(R^2+S^2); \quad (1)$$

while, $(P-iQ)/(R-iS)=(P-iQ)(R+iS)/(R+iS)(R-iS)$

$$=[(PR+QS)-i(QR-PS)]/(R^2+S^2). \quad (2)$$

Comparig (1), (2) we get the truth of the theorem.

Example 1. Prove that $(1+z+z^2)/(1-z+z^2)=(9+i6\sqrt{3})/7$ where $z=1-i\sqrt{3}$.

$1+z+z^2=1+(1-i\sqrt{3})+(1-i\sqrt{3})^2=2-i\sqrt{3}+1-2i\sqrt{3}-3=-3i\sqrt{3}$;

and similarly $1-z+z^2=-2-i\sqrt{3}$

$$\therefore \frac{1+z+z^2}{1-z+z^2}=\frac{-3i\sqrt{3}}{-2-i\sqrt{3}}=\frac{3i\sqrt{3}(2-i\sqrt{3})}{(2+i\sqrt{3})(2-i\sqrt{3})}=\frac{9+i6\sqrt{3}}{7}.$$

Example 2. If $a_0, a_1, a_2, \ldots a_{2n}$ are the coefficients in the expansion of $(1+x)^{2n}$ in ascending powers of x prove that

$$a_0+a_1-3(a_2+a_3)+9(a_4+a_5)-3^2(a_6+a_7)+3^4(a_8+a_9)$$

has the values $0, 2^{2n}, -2^{2n}$, according as the integer n is of the form $3m+1$, $3m$, $3m-1$. [*B. U 1917 Hons I.*]

By hypothesis

$$(1+x)^{2n}=a_0+a_1x+a_2x^2+a_3x^3+\ldots\ldots+a_{2n}x^{2n}.$$

Putting $x=i\sqrt{3}$ in this identity, we have

$$(1+i\sqrt{3})^{2n}=a_0+a_1i\sqrt{3}+a_2(i\sqrt{3})^2+a_3(i\sqrt{3})^3+\ldots\ldots$$

$$=a_0-3a_2+3^2a_4-\ldots$$

$$+i\sqrt{3}(a_1-3a_3+3^2a_5-\ldots).$$

Now $1+i\sqrt{3}=2(\cos\frac{1}{3}\pi+i\sin\frac{1}{3}\pi)$ [*Example* 4, § 3·4].

$$\therefore (1+i\sqrt{3})^{2n}=2^{2n}(\cos\tfrac{1}{3}\pi+i\sin\tfrac{1}{3}\pi)^{2n}=2^{2n}(\cos\tfrac{2}{3}n\pi+i\sin\tfrac{2}{3}n\pi).$$

Hence $2^{2n}\cos\frac{2}{3}n\pi+i2^{2n}\sin\frac{2}{3}n\pi$ (§ 3·8)

$$=(a_0-3a_2+3^2a_4-\ldots\ldots)+i\sqrt{3}(a_1-3a_3+3^2a_5-\ldots\ldots).$$

Equating the real and imaginary parts, we get

$$2^{2n}\cos n\tfrac{2}{3}\pi=a_0-3a_2+3^2a_4-\ldots\ldots \quad (1)$$

and $$2^{2n}\sin\tfrac{2}{3}n\pi=\sqrt{3}(a_1-3a_3+3^2a_5-\ldots\ldots) \quad (2)$$

Dividing (2) by $\sqrt{3}$ and then adding it to (1), we have

$$2^{2n}(\cos\tfrac{2}{3}n\pi+\tfrac{1}{\sqrt{3}}\sin\tfrac{2}{3}n\pi)=a_0+a_1-3(a_2+a_3)+3^2(a_4+a_5)-\ldots$$

From this, the required results follow immediately.

References.

A. N. Whitehead. 'An Introduction to Mathematics', Chapters, VI, VII, VIII.

J. W. Young. 'Lectures on Fundamental Concepts of Algebra and Geometry', lectures X, XI, XII.

G. H. Hardy 'Pure Mathematics', Chapter III.

J. W. Young. 'Monographs on Modern Mathematics,' 4th monograph. Only students of exceptional powers of abstraction should read the last book.

Examples. III. e.

1. Put the following into the form $X+iY$:

$(1+i)/(1-i)$; $(1+i)^2/(1-i)^2$; $(1+i)^3$; $(1-i)^3$; $(1-i)^2/(1+i)^4$, $(1+i)^4/(1-i)^4$, $(1+i)^4+(1-i)^4$. [Use $1+i=\sqrt{2}\,(\cos\frac{1}{4}\pi+i\sin\frac{1}{4}\pi)$.]

2. If $w=(-1+i\sqrt{3})/2$, prove the following :-

$w^3=1$, $w^2=w^{-1}=(-1-i\sqrt{3})/2$; $1+w+w^2=0$; $1+w^2+w^4=0$.

3. Find the modulus of

$$(1+i)(2+i)(1+i2) \text{ and of } (1-i)\ (2-i)(1-2i)$$

and verify that they are equal.

4. Find the modulus of

$$\frac{(1+2i)(3+i)(2+8i)}{(1+i4)(1+3i)(1+i)}.$$

5. Prove that $|x+iy|=x-iy|=\sqrt{(x+iy)(x-iy)}$.

6. Solve the quadratic equations:

(*a*). $x^2-(1+i)x+i=0$, (*b*) $x^2+ix+3-i=0$,

(*c*). $2x^2-4ix-1-i\sqrt{3}=0$, (*d*) $2x^2+i4x-3-i\sqrt{3}=0$.

7. If $lz_1+mz_2=0$ then $l=0=m$ where z_1, z_2 are two complex, and l and m are two real, numbers.

Deduce that if $lz_1+mz_2=l'z_1+m'z_2$, then $l=l'$ and $m=m'$ where, l, m, l', m' are real.

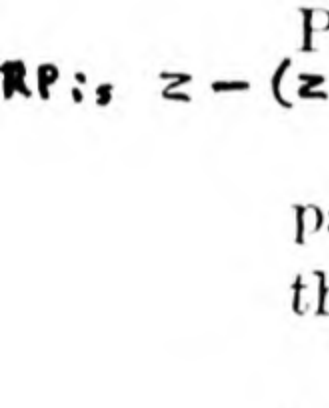
8. OAB is any $\triangle$. D, E divide OA, OB internally in the ratios 1:2 and 1:1 respectively. DE and AB intersect at F. Prove that $3DE=EF$ and $AB=BF$. [Let O be the origin in the Argand's diagram, and let $\overline{OD}$, $\overline{OA}$, $\overline{OE}$, $\overline{OB}$ represent vectors z_1, $3z_1$, z_2, $2z_2$ respectively. Suppose further that $DF=xDE$ and $AF=yAB$; then $\overline{OF}=\overline{OD}+\overline{DF}=z_1+x.\overline{DE}=z_1+x(\overline{OE}-\overline{OD})=z_1+x(z_2-z_1)$.

Similarly

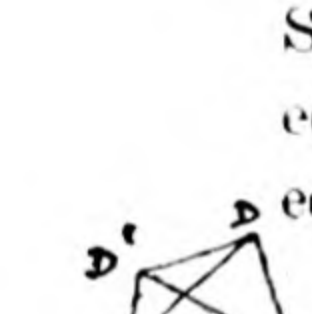
$\overline{OF}=\overline{OA}+\overline{AF}=3z_1+y(2z_2-3z_1)$; $\therefore z_1+x(z_2-z_1)=3z_1+y(2z_2-3z_1)$.

Equating the coefficients of z_1 and z_2 (see *Example* 7 above), we have $1-x=3-3y$, $x=2y$. etc.]

9. $z=lz_1+mz_2$, where $al+bm=1$, a, b, l, m being all real. Prove that z represents a point on the line connecting points representing z_1/a, z_2/b in the Argand's diagram.



10. $ODEF$ is a parallelogram. Points M, N are taken in FE so that MN is of constant length. Prove that OM, DN intersect on a st. line; so also ON, DM intersect on another.



11. $\triangle ODE$ is of fixed shape. The pt. O is fixed; while D moves along a fixed st. line. Prove that E moves along a fixed st. line.

12. G is the centroid of a $\triangle XYZ$. Starting from any point P, PQ, QR are drawn successively equal and parallel to GX, GY. Prove that RP is equal and parallel to GZ.

13. From any pt. O, OP, OQ, OR, are drawn equal and parallel to the sides MN, NL, LM of a $\triangle LMN$. Prove that O is the centroid of $\triangle PQR$.

14. G is the centroid and O any pt. in the plane of a $\triangle XYZ$. Starting from any pt. P, PQ, QR, RS are successively drawn equal and parallel to OX, OY, OZ. Prove that PS is parallel and equal to, three times OG.

15. $\triangle$s DEF, and $D'E'F'$ have the same centroid G.

I. Starting from any pt. P, PQ, QR, RS are successively drawn equal and parallel to DD', EE', FF'; prove that P and S coincide.

II. From any pt. O, OL, OM, ON are drawn equal and parallel to DD', EE', FF' respectively. Prove that O is the centroid of $\triangle LMN$.

CHAPTER IV.

DEMOIVRE'S THEOREM
AND
ITS CONSEQUENCES

4·01. This Chapter is devoted to DeMoivre's Theorem and its immediate applications. It is a theorem of great importance as the reader will realise in the course of the chapter. It provides a sort of bridge connecting the two important branches of Mathematics-Algebra and Trigonometry. By means of it, many algebraical identities are immediately converted into important results in Trigonometry. The passage from the real, to the complex, variable is also made easy by the same theorem. For this reason Chrystal describes it as "the powerful analytical engine."

Repetition 4·1. *Prove that*

$$(\cos A_1 + i\sin A_1)(\cos A_2 + i\sin A_2)\ldots(\cos A_n + i\sin A_n)$$
$$= (\cos A_1 + A_2 + \ldots + A_n) + i\sin(A_1 + A_2 + \ldots + A_n).$$

Since $i^2 = -1$, we have

$$(\cos A_1 + i\sin A_1)(\cos A_2 + i\sin A_2)$$
$$= (\cos A_1 \cos A_2 - \sin A_1 \sin A_2) + i\,(\sin A_1 \cos A_2 + \cos A_1 \sin A_2)$$
$$= \cos(A_1 + A_2) + i\sin(A_1 + A_2). \qquad (1)$$

Multiplying both sides of (1) by $(\cos A_3 + i\sin A_3)$, we have

$$(\cos A_1 + i\sin A_1)(\cos A_2 + i\sin A_2)(\cos A_3 + i\sin A_3)$$
$$= [\cos(A_1 + A_2) + i\sin(A_1 + A_2)](\cos A_3 + i\sin A_3)$$

Now by (1), $[\cos(A_1 + A_2) + i\sin(A_1 + A_2)](\cos A_3 + i\sin A_3)$
$$= [\cos(A_1 + A_2 + A_3) + i\sin(A_1 + A_2 + A_3)]$$

Thus $(\cos A_1 + i\sin A_1)(\cos A_2 + i\sin A_2)(\cos A_3 + i\sin A_3)$
$$= \cos(A_1 + A_2 + A_3) + i\sin(A_1 + A_2 + A_3).$$

Evidently this process may be repeated any number of times; and thus we prove the given identity.

4·2. DeMoivre's Theorem. *Prove that* $\cos nA+i\sin nA$ *is one of the values of* $(\cos A+i\sin A)^n$ *where* n *is any real number.*

Case I. Let n be a positive integer.

Putting $A_1=A_2=\ldots=A_n=A$ in the result of §4·1, we have immediately

$$(\cos A+i\sin A)^n=\cos nA+i\sin nA. \qquad (1)$$

Case II. Suppose n is a positive rational number p/q say, where p and q are positive integers. It is clear from (1) that

$$\left(\cos\frac{A}{q}+i\sin\frac{A}{q}\right)^q=\cos q\,\frac{A}{q}+i\sin q\,\frac{A}{q}=\cos A+i\sin A.$$

$$\therefore\ \cos\frac{A}{q}+i\sin\frac{A}{q}\ \text{is one of the values of}\ (\cos A+i\sin A)^{1/q}$$

Now raising $\cos\dfrac{A}{q}+i\sin\dfrac{A}{q}$ and $(\cos A+i\sin A)^{1/q}$ to the p^{th} power and observing that, p being a positive integer,

$$\left(\cos\frac{A}{q}+i\sin\frac{A}{q}\right)^p=\cos\frac{p}{q}A+i\sin\frac{p}{q}A,$$

we conclude that $\cos\dfrac{p}{q}A+i\sin\dfrac{p}{q}A$ is one of the values of

$$(\cos A+i\sin A)^{p/q}.$$

Thus the theorem is established for positive rational values of n.

Case III. Suppose n is equal to $-m$ where m is a positive rational numbere, so that by Case II, $\cos mA+i\sin mA$ is one of the values of $(\cos A+i\sin A)^m$; $\therefore (\cos mA+i\sin mA)^{-1}$ or $1/(\cos mA+i\sin mA)$ is one of the values of $(\cos A+i\sin A)^{-m}$.

$$\text{Now}\ \frac{1}{\cos mA+i\sin mA}=\frac{\cos mA-i\sin mA}{(\cos mA+i\sin mA)(\cos mA-i\sin mA)}$$

$$=\frac{\cos mA-i\sin mA}{\cos^2 mA+\sin^2 mA};$$

$$\therefore\ \frac{1}{\cos mA+i\sin mA}=\cos(-mA)+i\sin(-mA).$$

Hence,

$\cos(-mA)+i\sin(-mA)$ is one of the values of $(\cos A+i\sin A)^{-m}$.

Thus we see that when n is a positive or negative integer $\cos nA+i\sin nA$ is equal to $(\cos A+i\sin A)^n$, while if n is a non-in-

integral rational number, $\cos n.A+i\sin n.A$ is only one of the values of $(\cos.A+i\sin.A)^n$. The extension of the proof to irrational values of n is beyond the range of this book. Other values of $(\cos.A+i\sin.A)^n$ will be considered later on (§ 7·4).

4·21. The results of this article are very important. De Moivre's theorem operates through the medium of these results.

I. If $x=\cos A+i\sin A$ *and* n *is an integer, prove that*

(*a*) $x+x^{-1}=2\cos A$, *and* $x-x^{-1}=2i\sin A$,

and (*b*) $x^n+x^{-n}=2\cos nA$ *and* $x^n-x^{-n}=2i\sin nA$.

II. *Further, if* $x=\cos A+i\sin A$, $y=\cos B+i\sin B$ *and* m *and* n *are integers, prove that*

$x^m y^n+x^{-m}y^{-n}=2\cos(mA+nB)$ *and* $x^m y^n-x^{-m}y^{-n}=2i\sin(mA+nB)$.

I $x^{-1}=(\cos A+i\sin A)^{-1}=\cos(-A)+i\sin(-A)=\cos A-i\sin A$;

hence $x+x^{-1}=(\cos A+i\sin A)+(\cos A-i\sin A)=2\cos A$;

and $x-x^{-1}=(\cos A+i\sin A)-(\cos A-i\sin A)=2i\sin A$.

Farther by § 4·2, $x^n=\cos nA+i\sin nA$ and $x^{-n}=\cos nA-i\sin nA$.

Hence $x^n+x^{-n}=(\cos nA+i\sin nA)+(\cos nA-i\sin nA)=2\cos nA$,

and $x^n-x^{-n}=(\cos nA+i\sin nA)-(\cos nA-i\sin nA)=2i\sin nA$.

To prove II, we observe that, by § 4·2,

$$x^m=\cos mA+i\sin mA \text{ and } y^n=\cos nB+i\sin nB;$$

hence by § 4·1, $x^m y^n=(\cos mA+i\sin mA)(\cos nB+i\sin nB)$

$$=\cos(mA+nB)+i\sin(mA+nB).$$

Similarly $x^{-m}y^{-n}=(\cos mA-i\sin mA)(\cos nB-i\sin nB)$

$$=\cos(mA+nB)-i\sin(mA+nB).$$

Hence $x^m y^n+x^{-m}y^{-n}=2\cos(mA+nB)$

and $x^m y^n-x^{-m}y^{-n}=2i\sin(mA+nB)$.

It is clear that the result, II. may be further generalized thus;

If $x=\cos A+i\sin A$, $y=\cos B+i\sin B$, $z=\cos C+i\sin C$,....., and m, n, p,....., are integers, then

$$x^m y^n z^p\ldots+x^{-m}y^{-n}z^{-p}\ldots=2\cos(mA+nB+pC+\ldots)$$

and $$x^m y^n z^p\ldots-x^{-m}y^{-n}z^{-p}\ldots=2i\sin(mA+nB+pC+\ldots).$$

4·22. We shall consider now some of the important applications of §§ 4·1, 4·2.

Prove that

(*a*) $\cos(D_1+D_2+\ldots\ldots+D_n)$
$=\cos D_1\cos D_2\ldots\ldots\cos D_n(1-S_2+S_4-S_6+\ldots\ldots)$,

(*b*) $\sin(D_1+D_2+\ldots\ldots+D_n)$
$=\cos D_1\cos D_2\ldots\ldots\cos D_n(S_1-S_3+S_5-\ldots\ldots)$,

and (*c*) $\tan(D_1+D_2+\ldots\ldots+D_n)$
$=\dfrac{S_1-S_3+S_5-S_7+\ldots\ldots}{1-S_2+S_4-S_6+\ldots\ldots}$,

where S_r *stands for the sum of the products of the* n *quantities* $\tan D_1, \tan D_2, \ldots\ldots, \tan D_n$, *taken* r *at a time*

By § 4·1, we have the identity

$\cos(D_1+D_2+\ldots+D_n)+i\sin(D_1+D_2+\ldots+D_n)$
$=(\cos D_1+i\sin D_1)(\cos D_2+i\sin D_2)\ldots(\cos D_n+i\sin D_n)$

Now, $\cos D_r+i\sin D_r=\cos D_r(1+i\tan D_r)$, $(r=1,2,\ldots,n)$

and hence, $\cos(D_1+D_2+\ldots+D_n)+i\sin(D_1+D_2+\ldots+D_n)$
$=\cos D_1\cos D_2\ldots\cos D_n(1+i\tan D_1)(1+i\tan D_2)\ldots(1+i\tan D_n)$
$=\cos D_1\cos D_2\ldots\cos D_n(1+iS_1+i^2S_2+i^3S_3+\ldots+i^nS_n)$, (§ 1·2).

Now remembering that $i^2=-1$, $i^3=-i$, $i^4=1$, etc, we have

$\cos(D_1+D_2+\ldots+D_n)+i\sin(D_1+D_2+\ldots+D_n)$
$=\cos D_1\cos D_2\ldots\cos D_n\,[\,(1-S_2+S_4-S_6+\ldots)+i(S_1-S_3+S_5-\ldots)\,]$

Equating the real and imaginary parts, we get (*a*) and (*b*).

Dividing (*b*) by (*a*), we have (*c*).

The student should notice that (*a*), (*b*) and (*c*) are respectively the generalizations of the three elementary formulae;

(*a'*) $\cos(L_1\pm L_2)=\cos L_1\cos L_2(1\mp\tan L_1\tan L_2)$,
$=\cos L_1\cos L_2\mp\sin L_1\sin L_2$;

(*b'*) $\sin(L_1\pm L_2)=\cos L_1\cos L_2(\tan L_1\pm\tan L_2)$
$=\sin L_1\cos L_2\pm\cos L_1\sin L_2$;

and (*c'*) $\tan(L_1\pm L_2)=\dfrac{\tan L_1\pm\tan L_2}{1\mp\tan L_1\tan L_2}$.

4·23. *Prove that if* n *is a positive integer*

(I) $\cos nE=\cos^nE-{}_nC_2\cos^{n-2}E\sin^2E+{}_nC_4\cos^{n-4}E\sin^4E-\ldots$;

(II) $\sin nE={}_nC_1\cos^{n-1}E\sin E-{}_nC_3\cos^{n-3}E\sin^3E$
$+{}_nC_5\cos^{n-5}E\sin^5E-\ldots$;

and (III) $\tan nE=\dfrac{{}_nC_1\tan E-{}_nC_3\tan^3E+{}_nC_5\tan^5E-\ldots}{1-{}_nC_2\tan^2E+{}_nC_4\tan^4E-\ldots}$.

These results are immediately derived from §4·22, by setting $E_1 = E_2 = \ldots = E_n = E$. It is however more convenient to derive them directly from DeMoivre's theorem.

Since n is a positive integer, we have, by §4·2,

$$\cos nE + i\sin nE = (\cos E + i\sin E)^n.$$

Now by the Binomial theorem for positive integral index, we have

$(\cos E + i\sin E)^n = \cos^n E + i\,{}_nC_1\cos^{n-1}E\sin E + {}_nC_2 i^2\cos^{n-2}E\sin^2 E \ldots$

$= \cos^n E + i\,{}_nC_1\cos^{n-1}E\sin E - {}_nC_2\cos^{n-2}E\sin^2 E + \ldots.$

Thus, $\cos nE + i\sin nE = [\cos^n E - {}_nC_2\cos^{n-2}E\sin^2 E + {}_nC_4\cos^{n-4}E\sin^4 E - \ldots] + i[{}_nC_1\cos^{n-1}E\sin E - {}_nC_3\cos^{n-3}E\sin^3 E + \ldots]$.

Now equating real and imaginary parts, we get (II) and (I).

Dividing (II) by (I), we have

$$\tan nE = \frac{{}_nC_1\cos^{n-1}E\sin E - {}_nC_3\cos^{n-3}E\sin^3 E + {}_nC_5\cos^{n-5}E\sin^5 E - \ldots}{1 - {}_nC_2\cos^{n-2}E\sin^2 E + {}_nC_4\cos^{n-4}E\sin^4 E - \ldots}$$

Dividing the numerator and the denominator by $\cos^n E$, we get (III).

The following examples will bring home to the student, the truth of the remarks made, at the begining of the chapter, regarding DeMoivre's Theorem.

Example I. From the identity $\dfrac{1}{x-a} - \dfrac{1}{x-b} = \dfrac{a-b}{(x-a)(x-b)}$,

deduce

$$\cos(P+A)\sin(P-B) - \cos(P+B)\sin(P-A) = \sin(A-B)\cos 2P$$

$$\sin(P+A)\sin(P-B) - \sin(P+B)\sin(P-A) = \sin(A-B)\sin 2P,$$

[*B. U. 1924 Hons II.*]

Put x, a, b, respectively equal to

$\cos 2P + i\sin 2P$, $\cos 2A + i\sin 2A$, $\cos 2B + i\sin 2B$;

then by § 4·21, $\dfrac{1}{x-a} = \dfrac{x^{-\frac{1}{2}}a^{-\frac{1}{2}}}{x^{\frac{1}{2}}a^{-\frac{1}{2}} - a^{\frac{1}{2}}x^{-\frac{1}{2}}} = \dfrac{\cos(P+A) - i\sin(P+A)}{2i\sin(P-A)}$;

Similarly $\dfrac{1}{x-b} = \dfrac{\cos(P+B) - i\sin(P+B)}{2i\sin(P-B)}$,

and $\dfrac{a-b}{(x-a)(x-b)} = \dfrac{x^{-1}(a^{\frac{1}{2}}b^{-\frac{1}{2}} - b^{\frac{1}{2}}a^{-\frac{1}{2}})}{(x^{\frac{1}{2}}a^{-\frac{1}{2}} - a^{\frac{1}{2}}x^{-\frac{1}{2}})(x^{\frac{1}{2}}b^{-\frac{1}{2}} - x^{-\frac{1}{2}}b^{\frac{1}{2}})}$

$$= \frac{(\cos 2P - i\sin 2P)2i\sin(A-B)}{2i\sin(P-A)2i\sin(P-B)}.$$

Hence, we have $\frac{\cos(P+A)-i\sin(P+A)}{\sin(P-A)} - \frac{\cos(P+B)-i\sin(P+B)}{\sin(P-B)}$

$$= \frac{\sin(A-B)[\cos 2P \mp i\sin 2P]}{\sin(P-A)\sin(P-B)}.$$

Multiplying this identity by $\sin(P-A)\sin(P-B)$ and then equating real and imaginary parts, we get the required results.

Example 2. If a, b, c denote $\cos L+i\sin L$, $\cos M+i\sin M$, $\cos N+i\sin N$ respectively, express

$$P=\cos(M-N)+\cos(N-L)+\cos(L-M)$$

in terms of a, b, c. Show that if $P=-3/2$, then $\Sigma\cos L=\Sigma\sin L=0$ and that one set of angles satisfying $P=-3/2$ is given by $M=L+\frac{2}{3}\pi$ and $N=L-2\pi/3$. [*B. U.* 1922 *Hons* II.]

Now $b/c=\cos(M-N)+i\sin(M-N)$; and $c/b=\cos(M-N)-i\sin(M-N)$

$$\therefore\ b/c+c/b=2\cos(M-N)\ ;$$

Similarly $c/a+a/c=2\cos(N-L)$ and $a/b+b/a=2\cos(L-M)$.

Hence the relation, $P=-3/2$ transforms into

$$a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)+3abc=0$$

i. e. into $$(a+b+c)(bc+ca+ab)=0$$

Then either $$a+b+c=0,\ or\ bc+ca+ab=0.$$

If $a+b+c=0$, $(\cos L+i\sin L)+(\cos M+i\sin M)+(\cos N+i\sin N)=0$

$$\therefore\ \cos L+\cos M+\cos N=0=\sin L+\sin M+\sin N.$$

By eliminating N between these, we have

$$(\cos L+\cos M)^2+(\sin L+\sin M)^2=1 \text{ or } \cos(L-M)=-\tfrac{1}{2}\ :$$

$$\therefore\quad L-M=2r\pi\pm\tfrac{2}{3}\pi$$

Similarly $$L-N=2n\pi\pm\tfrac{2}{3}\pi \quad \text{and} \quad M-N=2m\pi\pm\tfrac{2}{3}\pi$$

where r, m, n are integers. ∴ It is easy to see that

$$M=L+\tfrac{2}{3}\pi,\ N=L-\tfrac{2}{3}\pi$$

is one set of values of angles satisfying $P=-3/2$.

Example 3. Sum the series (done in Inter.!)

$$\sin L+\sin(L+M)+\sin(L+2M)+\ldots \text{to } n \text{ terms.}$$

Denote the sum of the series by S, and $\cos L+i\sin L$ and $\cos M+i\sin M$, by a and b respectively. Then by §4·21,

$$2iS=(a-a^{-1})+(ab-a^{-1}b^{-1})+(ab^2-a^{-1}b^{-2})+\ldots$$

$$=a(1+b+b^2+\ldots\text{to } n \text{ terms})-a^{-1}(1+b^{-1}+b^{-2}+\ldots\text{to } n \text{ terms})$$

$$=a\frac{(1-b^n)}{1-b}-a^{-1}\frac{(1-b^{-n})}{1-b^{-1}}=ab^{\frac{1}{2}n-1}\frac{(b^{\frac{1}{2}n}-b^{\frac{1}{2}n})}{b^{-\frac{1}{2}}-b^{\frac{1}{2}}}-\frac{a^{-1}b^{\frac{1}{2}n-1}(b^{\frac{1}{2}n}-b^{-n\frac{1}{2}})}{b^{\frac{1}{2}}-b^{-\frac{1}{2}}}$$

Note: Summing G.P. of cplx. nos. cf. 11.84
No general statement in Bag i

$$= \frac{\sin\frac{1}{2}nM}{\sin\frac{1}{2}M}\left(ab^{\frac{1}{2}n-1} - a^{-1}b^{-\frac{1}{2}n-1}\right) = 2i\sin\left[L+\tfrac{1}{2}(n-1)M\right]\frac{\sin\frac{1}{2}nM}{\sin\frac{1}{2}M}$$

$$\therefore\ S = \frac{\sin nM/2}{\sin M/2}\sin\left[L+\tfrac{1}{2}(n-1)M\right].$$

Second method: Suppose that $C = \cos L + \cos(L+M) + \ldots$

then $C + iS = a + ab + ab^2 + \ldots + ab^{n-1} = a(1-b^n) \div (1-b)$.

$$= \frac{ab^{\frac{1}{2}n-1}\left(b^{\frac{1}{2}n} - b^{-\frac{1}{2}n}\right)}{b^{\frac{1}{2}} - b^{-\frac{1}{2}}} = \left[\cos\left(L+\tfrac{1}{2}(n-1)M\right) + i\sin\left(L+\tfrac{1}{2}(n-1)M\right)\right]\frac{\sin\frac{1}{2}nM}{\sin\frac{1}{2}M}$$

Now equating real and imaginary parts, we have immediately the sums of series C and S.

The student is advised to study both the methods of *Example* 3 above. In general, to sum a series

$$a_1\cos L + a_2\cos(L+M) + a_3\cos(L+2M) + \ldots$$

we choose $a_1\sin L + a_2\sin(L+M) + a_3\sin(L+2M) + \ldots$

as the companion series and vice versa. Then, in the notation of *Example* 3, above, we have

$$C + iS = a(a_1 + a_2b + a_3b^2 + \ldots + a_nb^{n-1}).$$

The latter is summed by the methods of algebra and the sum is exhibited in the form $P + iQ$ where P and Q are real. Equating the real and imaginary parts, we have the sums of the two series $C = P$ and $S = Q$. The student will find many instances of this method in later chapters.

Example 4. Prove that if n is a positive integer

$$\cos nX = 1 + 2n\sin\frac{X}{2}\cos\frac{X+\pi}{2} + \frac{n(n-1)}{1.2}2^2\sin^2\frac{X}{2}\cos 2\frac{(X+\pi)}{2} + \ldots$$

[*B. U. 1924 Hons II.*]

Denote the right hand series by C, and by S, the series

$$2n\sin\frac{X}{2}\sin\frac{X+\pi}{2} + \frac{n(n-1)}{1.2}2^2\sin^2\frac{X}{2}\sin 2\frac{(X+\pi)}{2} + \ldots$$

Then $C + iS = 1 + 2n\sin\frac{X}{2}\left(\cos\frac{X+\pi}{2} + i\sin\frac{X+\pi}{2}\right)$

$$+ \frac{n(n-1)}{1.2}2^2\sin^2\frac{X}{2}\left(\cos\frac{2(X+\pi)}{2} + i\sin\frac{2(X+\pi)}{2}\right) + \ldots\ldots$$

Now, if we denote $\cos\frac{X+\pi}{2}+i\sin\frac{X+\pi}{2}$ by z, we have, by § 4·2,

$$z^r=\cos\frac{r(X+\pi)}{2}+i\sin\frac{r(X+\pi)}{2}.$$

Hence $C+iS=1+n2\sin\frac{X}{2}.z+\frac{n(n-1)}{1.2}2^2\sin^2\frac{X}{2}z^2+\ldots$

$$=\left(1+2i\sin\frac{X}{2}z\right)^n$$

$$=\left[1+2\sin\tfrac{1}{2}X\left(\cos\frac{X+\pi}{2}+i\sin\frac{X+\pi}{2}\right)\right]^n$$

$$=\left[1-2\sin^2\frac{X}{2}+2i\sin\frac{X}{2}\cos\frac{X}{2}\right]^n$$

$$=(\cos X+i\sin X)^n=\cos nX+i\sin nX.$$

Therefore equating the real parts, we have $C=\cos nX$.

Example 5. By using the identity $(z^{m+1}-1)\div(z-1)=1+z+z^2+\ldots+z^m$, prove that

(a) $$\frac{r\sin A}{1-2r\cos A+r^2}=r\sin A+r^2\sin 2A+r^3\sin 3A+\ldots+r^m\sin mA$$
$$+\frac{r^{m+1}\sin(m+1)A-r^{m+2}\sin mA}{1-2r\cos A+r^2}$$

and (b) $$\frac{1-r^2}{1-2r\cos A+r^2}=1+2r\cos A+2r^2\cos 2A+\ldots+2r^m\cos mA$$
$$+\frac{2r^{m+1}\cos(m+1)A-2r^{m+2}\cos mA}{1-2r\cos A+r^2}.$$

Put $z=r\cos A+ir\sin A$ in the above identity; then we have

$$1+r(\cos A+i\sin A)+r^2(\cos 2A+i\sin 2A)+r^3(\cos 3A+i\sin 3A)+\ldots$$
$$=\frac{r^{m+1}[\cos(m+1)A+i\sin(m+1)A]-1}{r(\cos A+i\sin A)-1}. \qquad (1)$$

The right hand fraction of (1) is equal to

$$\frac{(r^{m+1}[\cos(m+1)A+i\sin(m+1)A]-1)[r(\cos A-i\sin A)-1]}{[r(\cos A+i\sin A)-1][r(\cos A-i\sin A)-1]}$$

or $$[r^{m+2}(\cos mA+i\sin mA)-r(\cos A-i\sin A)$$
$$-r^{m+1}(\cos(m+1)A+i\sin(m+1)A)+1]\div[r^2-2r\cos A+1]$$

$$\therefore\ 1+r\cos A+\ldots+r^m\cos mA+i(r\sin A+r^2\sin 2A+\ldots+r^m\sin mA)$$
$$=\frac{1-r\cos A-r^{m+1}\cos(m+1)A+r^{m+2}\cos mA}{1-2r\cos A+r^2}$$

$$+ i\,\frac{r\sin A - r^{m+1}\sin(m+1)A + r^{m+2}\sin mA}{1-2r\cos A + r^2}.$$

Equating real and imaginary parts and transforming the resulting equations we have (a), (b).

Example 6. Show that $\sum_{s=0}^{s=n}\sum_{r=0}^{r=m} \cos(rP+sQ)$ is equal to

$$\sin\tfrac{1}{2}(m+1)P \sin\tfrac{1}{2}(n+1)Q \operatorname{cosec}\tfrac{1}{2}P \operatorname{cosec}\tfrac{1}{2}Q \cos\tfrac{1}{2}(mP+nQ).$$

Denote the left hand series by C and the series

$$\sum_{s=0}^{s=n}\sum_{r=0}^{r=m} \sin(rP+sQ)$$

by S. If then, $x = \cos P + i\sin P$ and $y = \cos Q + i\sin Q$ we have

$$C+iS = \sum_{s=0}^{s=n}\sum_{r=0}^{r=m} x^r y^s$$

$$= (1+x+x^2+\ldots+x^m)(1+y+y^2+\ldots+y^n),$$

$$= \frac{x^{m+1}-1}{x-1}\cdot\frac{y^{n+1}-1}{y-1},$$

$$= \frac{x^{\frac{1}{2}m} y^{\frac{1}{2}n}\left(x^{\frac{1}{2}m+1} - x^{-\frac{1}{2}m+1}\right)\left(y^{\frac{1}{2}n+1} - y^{-\frac{1}{2}n+1}\right)}{\left(x^{\frac{1}{2}} - x^{-\frac{1}{2}}\right)\left(y^{\frac{1}{2}} - y^{-\frac{1}{2}}\right)}$$

$$= \left[\cos\tfrac{1}{2}(mP+nQ) + i\sin\tfrac{1}{2}(mP+nQ)\right] \frac{\sin\frac{1}{2}(m+1)P \sin\frac{1}{2}(n+1)Q}{\sin\frac{P}{2}\sin\frac{Q}{2}}$$

Now equating the real parts, we have the required result.

Examples IV. a.

1. Simplify $\dfrac{(\cos 5Y + i\sin 5Y)^3(\cos 2Y + i\sin 2Y)^5}{(\cos 2Y - i\sin 2Y)^8(\cos 3Y + i\sin 3Y)^9}$.

2. Simplify $\dfrac{(\cos 3Y + i\sin 3Y)^7(\cos Y - i\sin Y)^9}{(\cos 2Y + i\sin 2Y)(\cos 3Y - i\sin 3Y)^2}$.

3. Prove that
$(\cos Y + i\sin Y)(\cos 2Y + i\sin 2Y)(\cos 2^2 Y + i\sin 2^2 Y)\ldots$
$(\cos 2^{n-1} Y + i\sin 2^{n-1} Y) = \cos(2^n - 1)Y + i\sin(2^n - 1)Y.$

4. Prove that
$(\cos Y + i\sin Y)(\cos 2Y + i\sin 2Y)(\cos 3Y + i\sin 3Y)\ldots(\cos nY + i\sin nY)$
$= \cos mY + i\sin mY$ where $2m = n(n+1)$

3. If n is an integer, prove that

$$[1+\sin X+i\cos X]^n+[1+\sin X-i\cos X]^n = 2^{n+1}\cos^n\left(\frac{\pi}{4}-\frac{X}{2}\right)\cos\left(\frac{n\pi}{4}-\frac{nX}{2}\right)$$

6. From the identity $a^2(b^2-c^2)+b^2(c^2-a^2)+c^2(a^2-b^2)=0$, deduce that (1) $\Sigma\sin(B-C)\cos(2A+B+C)=0$ and (2) $\Sigma\sin(B-C)\sin(2A+B+C)=0$.

7. Deduce from the identity

$$(b^2-c^2)(a^2-x^2)+(c^2-a^2)(b^2-x^2)+(a^2-b^2)(c^2-x^2)=0,$$

$$\sin(B-C)\sin(A-P)+\sin(C-A)\sin(B-P)+\sin(A-B)\sin(C-P)=0.$$

8. From the identity

$$\frac{a^3}{(a-b)(a-c)(a-d)}+\frac{b^3}{(b-c)(b-d)(b-a)}+\frac{c^3}{(c-a)(c-b)(c-d)}+\frac{d^3}{(d-a)(d-b)(d-c)}=1,$$

deduce that $\Sigma\dfrac{\cos 2A}{\sin\dfrac{A-B}{2}\sin\dfrac{A-C}{2}\sin\dfrac{A-D}{2}}=8\sin\dfrac{A+B+C+D}{2}$

[*B. U. 1926 Hons. II*]

9. From the identity

$$a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2)=-(b-c)(c-a)(a-b)[bc+ca+ab]$$

deduce that

1. $\Sigma\cos 2P\sin(Q-R)$
$$=4\sin\frac{(Q-R)}{2}\sin\frac{R-P}{2}\sin\frac{P-Q}{2}[\Sigma\cos(Q+R)]$$

and 2. $\Sigma\sin 2P\sin(Q-R)$
$$=4\sin\frac{Q-R}{2}\sin\frac{R-P}{2}\sin\frac{P-Q}{2}[\Sigma\sin(Q+R)]$$

[*cf* Hobson's *Trigonometry*, § 38, (1). Smith's *conics*, §139.]

10. Prove the identity

$$(1+x)(1+x^2)(1+x^{2^2})\ldots\ldots(1+x^{2^{n-1}})=(1-x^{2^n})\div(1-x)$$

and deduce, by putting $x=\cos 2Y+i\sin 2Y$, that

$$2^n\cos Y\cos 2Y\cos 2^2Y\cos 2^3Y\ldots\ldots\cos 2^{n-1}Y=\sin 2^nY/\sin Y.$$

11. Prove the identity

$$(1-x+x^2)(1-x^2+x^4)(1-x^4+x^8)\ldots\ldots(1-x^{2^{n-1}}+x^{2^n})=\frac{1+x^{2^n}+x^{2^{n+1}}}{1+x+x^2},$$

and deduce that

$$(2\cos x-1)(2\cos 2x-1)(2\cos 2^2x-1)\ldots(2\cos 2^{n-1}x-1)$$
$$=(2\cos 2^n x+1)\div(2\cos x+1).$$

12. Prove that if n be a positive integer

$$\sin^n y\cos nx+n\sin^{n-1}y\cos(n-1)x\sin(x-y)+$$
$$+\frac{n(n-1)}{2}\sin^{n-2}y\cos(n-2)x\sin^2(x-y)+\ldots+\sin^n(x-y)=\sin^n y\cos ny.$$

[*B. U. 1926 Hons. II.*]

13. Prove that

$$\cos(A+B+C)+\cos(-A+B+C)+\cos(A-B+C)$$
$$+\cos(A+B-C)=4\cos A\cos B\cos C.$$

14. Prove that

$$-\sin(A+B+C)+\sin(-A+B+C)+\sin(A-B+C)$$
$$+\sin(A+B-C)=4\sin A\sin B\sin C.$$

15. Deduce from §4·23, (I) that if n is even $\cos nE$ can be expanded into a polynomial of even powers of $\cos E$ and that the coefficient of $\cos^n E$ is 2^{n-1}. Prove also in the same case that $\cos nE$, can be expanded into a polynomial of even powers of $\sin E$ and that the coefficient of $\sin^n E$ is $(-1)^{\frac{1}{2}n}2^{n-1}$. Deduce a similar result when n is odd. *e.g.*

$$\cos 4E=2^3\cos^4 E-8\cos^2 E+1=2^3\sin^4 E-8\sin^2 E+1.$$
$$\cos 5E=2^4\cos^5 E-20\cos^3 E+5\cos E.$$

(II) Deduce similarly that if n is even, $\sin nE/\sin E$ can be expanded into a polynomial in odd powers of $\cos E$ and that the coefficient of $\cos^{n-1}E$ is 2^{n-1}. Again show that if n is odd, $\sin nE/\sin E$ can be expanded into a polynomial of even powers of $\cos E$, the coefficient of the highest power being as before 2^{n-1}.

e. g.
$$\sin 7E/\sin E=2^6\cos^6 E-80\cos^4 E+24\cos^2 E-1,$$
and
$$\sin 6E/\sin E=2^5\cos^5 E-32\cos^3 E+6\cos E.$$

4·3. The repeated integration and differentiation of a trigonometrical function consisting of a sum of products of sines and cosines of an angle, are very much facilitated by expressing the function in the form of a sum of sines and cosines of multiples of

the angle. The following articles show how the transformation can be effected.

Express $\cos^n A$ as the sum of cosines of multiples of A, n being a positive integer.

Denoting $\cos A + i\sin A$ by a, we have, by § 4·21,

$$2^n\cos^n A = (a+a^{-1})^n$$
$$= a^n + {}_nC_1 a^{n-1}(a^{-1})^1 + {}_nC_2 a^{n-2}(a^{-1})^2 + {}_nC_3 a^{n-3}(a^{-1})^3 + \ldots$$
$$= a^n + {}_nC_1 a^{n-2} + {}_nC_3 a^{n-4} + \ldots + {}_nC_{n-1} a^{-n+2} + {}_nC_n a^{-n}. \quad (1)$$

Now remembering that ${}_nC_0 = {}_nC_n$, ${}_nC_1 = {}_nC_{n-1}$, ${}_nC_2 = {}_nC_{n-2}$ etc, we have

$$2^n\cos^n A = (a^n + a^{-n}) + {}_nC_1(a^{n-2} + a^{-n+2}) + {}_nC_2(a^{n-4} + a^{-n+4}) + \ldots$$

By § 4·21, $a^n + a^{-n} = 2\cos nA$, $a^{n-2} + a^{-n+2} = 2\cos(n-2)A \ldots$

$$\therefore\ 2^n\cos^n A = 2\cos nA + {}_nC_1\, 2\cos(n-2)A + {}_nC_2\, 2\cos(n-4)A + \ldots \quad (2)$$

If n be even, (1) contains $n+1$ terms, the middle term of which is the $\frac{1}{2}n+1$ th term. Hence it is equal to ${}_nC_{\frac{1}{2}n} a^{-\frac{1}{2}n} a^{\frac{1}{2}n}$ or ${}_nC_{\frac{1}{2}n}$.

Thus if n is even, the last term in (2) is

$$n! \div [(\tfrac{1}{2}n)!\,(\tfrac{1}{2}n)!].$$

If n be odd, (1) contains an even number of terms. So that the series has two middle terms, the $\frac{1}{2}(n+1)$th and $\frac{1}{2}(n+3)$th. These terms are ${}_nC_{\frac{1}{2}(n-1)} a^{+1}$ and ${}_nC_{\frac{1}{2}(n-1)} a^{-1}$. The sum of these two terms is

${}_nC_{\frac{1}{2}(n-1)}\ 2\cos A$ which is, in this case, the last term in (2).

4·31. The corresponding expansion for $\sin^n A$ is derived from that of $\cos^n A$ by substitnting $\frac{1}{2}\pi + A$ for A. Thus if n is even, we nave from § 4·3,

$$2^n\cos^n(\tfrac{1}{2}\pi + A) = 2\cos n(\tfrac{1}{2}\pi + A) + {}_nC_1 2\cos(n-2)(\tfrac{1}{2}\pi + A) + \ldots \quad (1)$$

Now $2^n\cos^n(\frac{1}{2}\pi + A) = (-1)^n 2^n \sin^n A$,

$$2\cos n(\tfrac{1}{2}\pi + A) = 2(-1)^{\frac{1}{2}n}\cos nA$$

$$2\cos(n-2)(\tfrac{1}{2}\pi + A) = 2(-1)^{\frac{1}{2}(n-1)}\cos(n-2)A, \text{ etc.}$$

Hence if n is even, (1) reduces to

$$2^n\sin^n A = (-1)^{\frac{1}{2}n}[2\cos nA - {}_nC_1 2\cos(n-2)A + {}_nC_2 2\cos(n-4)A - \ldots$$

If n is odd, $2^n\cos^n(\frac{1}{2}\pi + A) = -2^n\sin^n A$

$$2\cos n(\tfrac{1}{2}\pi + A) = 2(-1)^{\frac{1}{2}(n+1)}\sin nA$$

$2\cos[(n-2)(\frac{1}{2}\pi+A)]=2(-1)^{\frac{1}{2}n-1}\sin(n-2)A$, etc.

and hence in this case (1) reduces to

$$2^n\sin^n A=(-1)^{\frac{1}{2}n-1}[2\sin nA-{}_nC_1 2\sin(n-2)A+{}_nC_2 2\sin(n-4)A-\ldots]$$

An independent investigation of the above results is simple and quite similar to that of § 4·3.

The following examples point out some new applications of the above method.

Example 1. Express $\sin^4X\cos^5X$ in terms of [sines and] cosines of multiples of X.

Suppose $x=\cos X+i\sin X$; then by § 4·21, $x-x^{-1}=2i\sin X$, $x+x^{-1}=2\cos X$ and $x^r+x^{-r}=2\cos rX$ where r is an integer.

$$\therefore\ (2i\sin X)^4(2\cos X)^5=(x-x^{-1})^4(x+x^{-1})^5=(x^2-x^{-2})^4(x+x^{-1})$$
$$=(x^8-4x^4+6-4x^{-4}+x^{-8})(x+x^{-1})$$
$$=(x^9+x^{-9})+(x^7+x^{-7})-4(x^5+x^{-5})-4(x^3+x^{-3})+6(x+x^{-1})$$
$$=2\cos 9X+2\cos 7X-8\cos 5X-8\cos 3X+12\cos X$$
$$\therefore\ \sin^4X\cos^5X=2^{-8}(\cos 9X+\cos 7X-4\cos 5X-4\cos 3X+6\cos X).$$

Example 2, Express $\cos A\cos B\cos C$ as the sum of sines of angles of the form $pA+qB+rC$ where p, q, r are integers, ±1

Denote $\cos A+i\sin A$, $\cos B+i\sin B$ and $\cos C+i\sin C$ by a,b,c; then by §4·21, $a+a^{-1}=2\cos A$, etc.

$$\therefore 2^3\cos A\cos B\cos C=(a+a^{-1})(b+b^{-1})(c+c^{-1})$$
$$=[(abc+a^{-1}b^{-1}c^{-1})+(a^{-1}bc+ab^{-1}c^{-1})+(ab^{-1}c+a^{-1}bc^{-1})$$
$$+(abc^{-1}+a^{-1}b^{-1}c)]$$

Now by §4·21, $abc+a^{-1}b^{-1}c^{-1}=2\cos(A+B+C)$;

$a^{-1}bc+ab^{-1}c^{-1}=2\cos(-A+B+C)$, etc.

Hence $8\cos A\cos B\cos C$ is equal to

$$2\cos(A+B+C)+2\cos(-A+B+C)+2\cos(A-B+C)$$
$$+2\cos(A+B-C).$$

$$\therefore\ \cos A\cos B\cos C=\tfrac{1}{4}[\cos(A+B+C)+\cos(-A+B+C)$$
$$+\cos(A-B+C)+\cos(A+B-C)]$$

Example 3. Show that when m and n are positive integers the sum of the series

$$\cos^n x+\cos^n(x+2\pi/m)+\cos^n(x+4\pi/m)+\cos^n(x+6\pi/m)$$
$$+\ldots+\cos^n[x+2(m-1)\pi/m]$$

is independent of x. [*B. U. 1919 Hons* II.]

By § 4·3, we have the identity

$$2^n\cos^n A=2\cos nA+{}_nC_1 2\cos(n-2)A+{}_nC_2 2\cos(n-4)A+\ldots$$

Putting successively

$$A=x,\ x+2\pi/m,\ x+4\pi/m,\ x+6\pi/m\ldots x+2(m-1)\pi/m,$$

we have $2^n\cos^n x=2\cos nx+2\,{}_nC_1\cos(n-2)x+{}_nC_2 2\cos(n-4)x+\ldots$

$2^n\cos^n(x+2\pi/m)=2\cos n(x+2\pi/m)+{}_nC_1 2\cos(n-2)(x+2\pi/m)+\ldots,$

$2^n\cos^n(x+4\pi/m)=2\cos n(x+4\pi/m)+{}_nC_1 2\cos(n-2)(x+4\pi/m)+\ldots,$

.....

Adding up these equations and observing that

$$\sum_{r=0}^{r=m-1}\cos k(x+2r\pi/m)=0$$

where k is is an integer and that the last term in each of the series is ${}_nC_{\frac{1}{2}n}$ we have if n is even

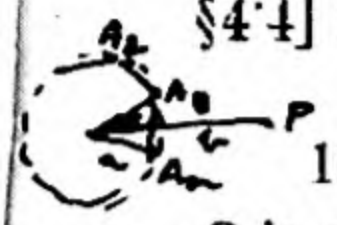
$$2^n S=m\,{}_nC_{\frac{1}{2}n} \text{ or } S=m2^{-n}\frac{n!}{\frac{1}{2}n!\,\frac{1}{2}n!}.$$

and if n is odd $2^n S=0$ the last term in this case being ${}_nC_{\frac{1}{2}n-\frac{1}{2}}2\cos x$.

Examples. IV. b.

Prove the following identities:

1. $16\sin^5 x=\sin 5x-5\sin 3x+10\sin x$;
2. $32\sin^6 x=10-15\cos 2x+6\cos 4x-\cos 6x$;
3. $64\sin^7 x=35\sin x-21\sin 3x+7\sin 5x-\sin 7x$;
4. $64\sin^4 x\cos^3 x=\cos 7x-\cos 5x-3\cos 3x+3\cos x$;
5. $64\cos^4 x\sin^3 x=3\sin x+3\sin 3x-\sin 5x-\sin 7x$,
6. $2^7\sin^8(x+\frac{1}{4}\pi)=2^7\cos^8(\frac{1}{4}\pi-x)=\cos 8x-8\sin 6x-28\cos 4x+56\sin 2x+70$
7. $2^2\sin A\sin B\sin C=\sin(-A+B+C)+\sin(A-B+C)$
$+\sin(A+B-C)-\sin(A+B+C)$,
8. $8\sin^2 A\sin^2 B=\cos(2A+2B)+\cos(2A-2B)-2\cos 2A-2\cos 2B+2$,
9. $8\cos^2 2x\sin^2 x=-(\cos 6x-2\cos 4x+3\cos 2x-2)$.



10. $A_1, A_2, A_3 \ldots A_m$ are the vertices of a regular polygon and P is any point in the plane of the polygon. Prove that

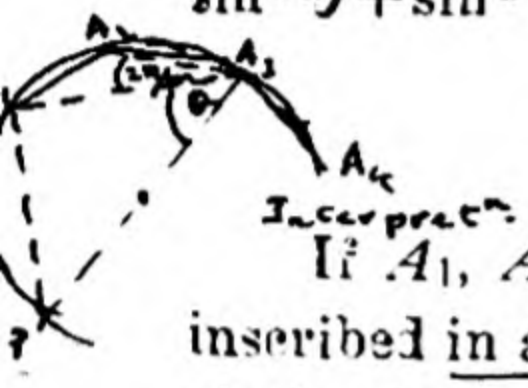
(I) $PA_1^4 + PA_2^4 + \ldots + PA_m^4 = m(a^4 + 2^2 a^2 b^2 + b^4)$

(II) $PA_1^6 + PA_2^6 + \ldots + PA_m^6 = m(a^6 + 3^2 a^4 b^2 + 3^2 a^2 b^4 + b^6)$

a being the radius of the circumcircle of the polygon and b the distance of P from its centre.

11. Prove that if m and n are integers, the sum

$$\sin^{2n}\theta + \sin^{2n}(\theta + \pi/m) + \sin^{2n}(\theta + 2\pi/m) + \ldots + \sin^{2n}(\theta + m-1\pi/m,) = m \cdot 2^{-2n} \cdot \frac{(2n)!}{n!\,n!}.$$



If $A_1, A_2, A_3, \ldots A_m$ are the vertices of a regular polygon inscribed in a circle and P is any point on the circumference of the circle, shew that

$$PA_1^{2n} + PA_2^{2n} + PA_3^{2n} + \ldots + PA_m^{2n} = m R^{2n} \frac{2n!}{n!\,n!}$$

R being the radius of the circle.

12. $P_1, P_2, P_3 \ldots, P_m$ are points on an ellipse with semiaxes a and b. If the eccentric angles of these are $\theta, \theta + 2\pi/m, \theta + 4\pi/m, \ldots, \theta + 2(m-1)\pi/m$, prove that

$$SP_1^4 + SP_2^4 + SP_3^4 + \ldots + SP_m^4 = S'P_1^4 + S'P_2^4 + \ldots + S'P_m^4 = (35a^4 - 30a^2b^2 + 3b^4)/8$$

where S and S' are the foci of the ellipse.

4·4. We now consider the inverse problem of expanding $\cos nx$ and $\sin nx$ as polynomials in $\cos x$ or $\sin x$. The several results are immediately derived from the algebraic identities established in §§ 1·5, 1·6.

To expand $\sin nx/\sin x$ into a series of descending powers of $\cos x$, we put $a = \cos x + i\sin x$, $b = \cos x - i\sin x$ in (1) of § 1·5. Now it is easily seen that $p = 2\cos x$, $q = 1$ and by § 4·21, $a^r - b^r = 2i\sin rx$ and $a^r + b^r = 2\cos rx$ where r is an integer. Making these substitutions in (1) of § 1·5, we have

$$\sin nx/\sin x = (2\cos x)^{n-1} - {}_{n-2}C_1(2\cos x)^{n-3} + {}_{n-3}C_2(2\cos x)^{n-5} - {}_{n-4}C_3(2\cos x)^{n-7} + \ldots + (-1)^r {}_{n-r-1}C_r(2\cos x)^{n-2r-1} + \ldots \quad (1)$$

Similarly from (2), (3) of § 1·5, we have expansions for

$\sin nx \div \sin x$ in ascending powers of $\cos x$. Thus if n is even,

$$(-1)^{\frac{1}{2}n-1}\frac{\sin nx}{\sin x}=\frac{n}{1!}\cos x-\frac{n(n^2-2^2)}{3!}\cos^3 x+\frac{n(n^2-2^2)(n^2-4^2)}{5!}\cos^5 x-$$

$$\ldots+(-1)^{r-1}\frac{n(n^2-2^2)(n^2-4^2)\ldots(n^2-\overline{2r-2}^2)}{(2r-1)!}\cos^{2r-1}x+\ldots\ldots(2)$$

and if n is odd,

$$(-1)^{\frac{1}{2}\overline{n-1}}\frac{\sin nx}{\sin x}=1-\frac{n^2-1^2}{2!}\cos^2 x+\frac{(n^2-1^2)(n^2-3^2)}{4!}\cos^4 x-\ldots\ldots$$

$$+(-1)^r\frac{(n^2-1^2)(n^2-3^2)(n^2-5^2)\ldots(n^2-\overline{2r-1}^2)}{(2r)!}\cos^{2r}x+\ldots\ldots\ (3)$$

From (1), (2), (3), of 1·6, we have expansions for $\cos nx$ in descending and ascending powers of $\cos x$.

Thus from (1) of § 1·6, we have

$$\cos nx=2^{n-1}\cos^n x-\frac{n}{1!}2^{n-3}\cos^{n-2}x+\frac{n(n-3)}{2!}2^{n-5}\cos^{n-4}x-\ldots$$

$$+(-1)^r\frac{n(n-r-1)(n-r-2)\ldots(n-2r+1)}{1.2.3.4\ldots r}2^{n-2r-1}\cos^{n-2r}x+\ldots,\ (4)$$

From (2) of § 1·6, we have, if n is even,

$$(-1)^{\frac{1}{2}n}\cos nx=1-\frac{n^2}{2!}\cos^2 x+\frac{n^2(n^2-2^2)}{4!}\cos^4 x-\ldots$$

$$+(-1)^r\frac{n^2(n^2-2^2)(n^2-4^2)\ldots(n^2-\overline{2r-2}^2)}{(2r)!}\cos^{2r}x\ldots\ldots\qquad(5)$$

and if n is odd, we have from (3) of § 1·6,

$$(-1)^{\frac{1}{2}\overline{n-1}}\cos nx=\frac{n}{1!}\cos x-\frac{n(n^2-1^2)}{3!}\cos^3 x+\frac{n(n^2-1^2)(n^2-3^2)}{5!}\cos^5 x-$$

$$\ldots+(-1)^r\frac{n(n^2-1^2)(n^2-3^2)(n^2-5^2)\ldots(n^2-\overline{2r-1}^2)}{(2r+1)!}\cos^{2r+1}x+\ldots\ (6)$$

N.B. It is interesting to see how (4), (5), (6) may be obtained from (1), (2), (3) respectively by first multiplying them by $\sin x$ and then integrating them between $\frac{1}{2}\pi$ and x. The remark saves the student nearly half the trouble.

From the last six formulae, we derive eight more by substituting $\frac{1}{2}\pi-x$ for x in them. Thus, observing that $\sin n(\frac{1}{2}\pi-x)$ is $(-1)^{\frac{1}{2}n-1}\sin nx$ or $(-1)^{\frac{1}{2}\overline{n-1}}\cos nx$ according as n is even or odd, we have from (1), if n is even,

$$(-1)^{\frac{1}{2}n-1}\frac{\sin nx}{\cos x}=(2\sin x)^{n-1}-\frac{n-2}{1}(2\sin x)^{n-3}+\frac{(n-3)(n-4)}{2!}(2\sin x)^{n-5}+\dots \qquad (7)$$

and if n is odd

$$(-1)^{\frac{1}{2}n-\frac{1}{2}}\frac{\cos nx}{\cos x}=(2\sin x)^{n-1}-\frac{n-2}{1!}(2\sin x)^{n-3}+\frac{(n-3)(n-4)}{2!}(2\sin x)^{n-5}\dots \qquad (8)$$

Similarly from (2) and (3) we have if n is even

$$\frac{\sin nx}{\cos x}=\frac{n}{1!}\sin x-\frac{n(n^2-2^2)}{3!}\sin^3 x+\frac{n(n^2-2^2)(n^2-4^2)}{5!}\sin^5 x-\dots \qquad (9)$$

and if n is odd

$$\frac{\cos n\,x}{\cos x}=1-\frac{n^2-1^2}{2!}\sin^2 x+\frac{(n^2-1^2)(n^2-3^2)}{4!}\sin^4 x-\dots \qquad (10)$$

In the same way, from (4), we have if n is even,

$$(-1)^{\frac{1}{2}n}\cos nx=2^{n-1}\sin^n x-\frac{n}{1!}2^{n-3}\sin^{n-2}x+\frac{n(n-3)}{2!}2^{n-5}\sin^{n-4}x-\dots \qquad (11)$$

and if n is odd,

$$(-1)^{\frac{1}{2}n-\frac{1}{2}}\sin nx=2^{n-1}\sin^n x-\frac{n}{1!}2^{n-3}\sin^{n-2}x+\frac{n(n-3)}{2!}2^{n-5}\sin^{n-4}x-\dots \qquad (12)$$

Similary from (5,) we have if n is even,

$$\cos nx=1-\frac{n^2}{2!}\sin^2 x+\frac{n^2(n^2-2^2)}{4!}\sin^4 x-\frac{n^2(n^2-2^2)(n^2-4^2)}{6!}\sin^6 x\dots \qquad (13)$$

and if n is odd, we have, from (6),

$$\sin nx=\frac{n}{1!}\sin x-\frac{n(n^2-1^2)}{3!}\sin^3 x+\frac{n(n^2-1^2)(n^2-3^2)}{5!}\sin^5 x-\dots \qquad (14)$$

These formulae are generalizations of the elementary formulae:

$$\sin 2x=2\sin x\cos x,\ \sin 3x/\sin x=3-4\sin^2 x=4\cos^2 x-1,$$
$$\cos 2x=2\cos^2 x-1=1-2\sin^2 x,\ \cos 3x=4\cos^3 x-3\cos x.$$

We may obtain the above formulae by using the series of *Example* 5, §4·23. Other methods will be suggested later on.

Example 1. Prove that

(*a*) $2\cos 5x=(2\cos x)^5-5(2\cos x)^3+5(2\cos x)$;

and (*b*) $\sin 5x = \sin x(1-12\cos^2 x+16\cos^4 x)$,

Using formula (4) above, we have

$$\cos 5x = 2^{5-1}\cos^5 x - \frac{5}{1!}2^{5-3}\cos^3 x + \frac{5.2}{2!}2^{5-5}\cos x$$

$$= 2^4\cos^5 x - 5.2^2\cos^3 x + 5\cos x$$

$$\therefore 2\cos 5x = (2\cos x)^5 - 5(2\cos x)^3 + 5(2\cos x).$$

To obtain (*b*), we observe that 5 is odd; hence using (3) above, we have

$$\sin 5x/\sin x = 1 - \frac{5^2-1^2}{2!}\cos^2 x + \frac{(5^2-1^2)(5^2-3^2)}{4!}\cos^4 x$$

$$\therefore \sin 5x = \sin x(1-12\cos^2 x+16\cos^4 x).$$

Example 2. In any $\triangle ABC$, prove that

$$a^n\cos nB + b^n\cos nA = c^n - nabc^{n-2}\cos(A-B) + \frac{n(n-3)}{2!}(ab)^2c^{n-4}\cos 2(A-B) - \ldots ;$$

$$a^n\sin nB - b^n\sin nA = nabc^{n-2}\sin(A-B) - \frac{n(n-3)}{2!}(ab)^2c^{n-4}\sin 2(A-B) + \ldots$$

where the coefficients in the two series are the same as those in the expansion of $2\cos nx$ in descending powers of $2\cos x$

[*B. U. 1925 Problems.*]

Denoting the two series respectively by C and S and $\cos A + i\sin A$ and $\cos B + i\sin B$ by x and y, we have

$$C + iS = c^n - nabc^{n-2}x^{-1}y^{-1} + \frac{n(n-3)}{2!}a^2b^2x^{-2}y^2 - \ldots\ldots \quad (1)$$

Now since $a\cos B + b\cos A = c$ and $a\sin B - b\sin A = 0$, we have

$$c = a\cos B + b\cos A + a\,i\sin B - bi\sin A$$

$$= a(\cos B + i\sin B) + b(\cos A - i\sin A) = ay + bx^{-1}.$$

Hence (1) may be written in the form

$$C + iS = (ay+bx^{-1})^n - n\,aybx^{-1}(ay+bx^{-1})^{n-2} + \frac{n(n-3)}{2!}(ay)^2(bx^{-1})^2(ay+bx^{-1})^{n-4} + \ldots \quad (2)$$

Comparing (2) with the formula (1) of § 1·6, we have

$$C + iS = a^n y^n + b^n x^{-n} = a^n(\cos nB + i\sin nB) + b^n(\cos nA - i\sin nA)$$

$$= a^n\cos nB + b^n\cos nA + i(a^n\sin nB - b^n\sin nA).$$

Equating the real and imaginary parts, we get the required results.

Examples IV. c.

Prove the following identities:–

1. $\sin 7a = 7\sin a - 56\sin^3 a + 112\sin^5 a - 64\sin^7 a$;
2. $\cos 9a = 2^8\cos^9 a - 9.2^6\cos^7 a + 27.2^4\cos^5 a - 30.2^2\cos^3 a + 9\cos a$;
3. $\sin 6a = \sin a[\ (2\cos a)^5 - 4(2\cos a)^3 + 3(2\cos a)\ ]$.

If $x = 2\cos y$, then prove the following identities:

4. $2\cos 7y - 2\cos 3y = x(x^2-4)(x^4-3x^2+1)$;
5. $2\cos 7y - 2\cos y = x(x^2-1)(x^2-2)(x^2-4)$;
6. $2\cos 6y - 2\cos 2y = x^2(x^2-2)(x^2-4)$;
7. $2\cos 8y - 2\cos 4y = x^2(x^2-1)(x^2-3)(x^2-4)$;
8. $2\cos 7y - 2\cos 2y = (x-2)(x^2+x-1)(x+1)(x^3-3x+1)$;
9. $2 + 2\cos 8y = (x^4-4x^2+2)^2$.

10. Show that

$$\sin na\sin x + \sin(n-1)a\sin 2x + \sin(n-2)a\sin 3x + \ldots + \sin a\sin nx = \frac{\sin a\sin(n+1)x - \sin(n+1)a\sin x}{2(\cos x - \cos a)}.$$

Hence or otherwise sum the series

$$n\sin x + (n-1)\sin 2x + (n-2)\sin 3x + \ldots\ldots + 2\sin(n-1)x + \sin nx$$

and $$n\cos x + (n-1)\cos 2x + (n-2)\cos 3x + \ldots\ldots + 2\cos(n-1)x + \cos nx$$

[*B. U. 1923 Problems.*]

11. Prove that if n is odd

$$[\ 2\cos(n-1)a + 2\cos(n-3)a + 2\cos(n-5)a + \ldots + 2\cos 2a + 1\ ]$$
$$\times[2\cos(n-1)a - 2\cos(n-3)a + 2\cos(n-5)a + \ldots\ldots + (-1)^{\frac{1}{2}(n-1)}\ ]$$
$$= [\ 2\cos 2(n-1)a + 2\cos 2(n-3)a + 2\cos 2(n-5)a + \ldots + 2\cos 4a + 1\ ].$$

12. Prove that in a $\triangle ABC$,

$$a^n\cos nC = b^n - {}_nC_1 b^{n-1}c\cos A + {}_nC_2 b^{n-2}c^2\cos 2A - \ldots\ldots;$$
and $$a^n\sin nC = {}_nC_1 b^{n-1}c\sin A - {}_nC_2 b^{n-2}c^2\sin 2A + {}_nC_3 b^{n-3}c^3\sin 3A + \ldots$$

13. Prove that, in a $\triangle ABC$,

$$a^n\cos nX = b^n\cos Y - {}_nC_1 b^{n-1}c\cos(Y-A) + {}_nC_2 b^{n-2}c^2\cos(Y-2A) - \ldots$$
$$a^n\sin nX = b^n\sin Y - b^{n-1}c\sin(Y-A) + {}_nC_2 b^{n-2}c^2\sin(Y-2A) - \ldots$$

where $Y = nX - nC$, X being any angle.

14. P, Q, R,...are the vertices of a regular polygon inscribed in a circle of redius a; and O is any pt. in the plane of the circle, at a distance b from the centre. Prove that

$$O\,P^{2n}+OQ^{2n}+OR^{2n}+\ldots\ldots$$
$$=n[a^{2n}+({}_nC_1)^2a^{2n-2}b^2+({}_nC_2)^2a^{2n-4}b^4+({}_nC_3)^2a^{2n-6}b^6+\ldots]$$

15. Prove that

$$2^{n-1}\cos A\cos B\cos C\cos D\ldots=\Sigma\cos(A\pm B\pm C\pm D\ldots)$$

where the sum includes all terms corresponding to all possible variations of sign.

[Put $2\cos A=a+a^{-1}$, $2\cos B=b+b^{-1}$ etc. and use § 4·21.

e. g. $2\cos A\cos B=\cos(A+B)+\cos(A-B)$.]

16. If $(a_1+ib_1)(a_2+ib_2)\ldots(a_n+ib_n)=A+iB$, where a_1, a_2,... $b_1, b_2, \ldots\ldots, A$, B are real, prove that

$$\tan^{-1}\frac{b_1}{a_1}+\tan^{-1}\frac{b_2}{a_2}+\tan^{-1}\frac{b_3}{a_3}+\ldots=\tan^{-1}\frac{B}{A}$$

~~to a multiple of 2π~~.

Solⁿ of quadratics implicit in Euc. (Ball 56) — C.B.M.P. I 270
250 Arabian quadratic
1050 " classif^n of cubics (geom^l.)
1500 Nicomachus (Farrar)
Tartaglia
Cardan
1600
1700
1800

CHAPTER V

ALGEBRAIC EQUATIONS (PART I)

1900

5·0. An expression of the form

$$f(x) \equiv p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_{n-1}x + p_n$$

which is obtained by carrying out, a finite number of times, the algebraic operations of addition, subtraction, and multiplication upon a quantity x, is called a rational integral function of x, or a polynomial in x. It is said to be of the n th degree, n being the exponent of the highest power of x. *e.g.* x^5+x-1, $x^2+\sqrt{2}$, $x+i\sqrt{2}$, are polynomials while $x^{\frac{2}{3}}+1$ is not.

An exclusive def^n ? Worth while Q. Hardy's inaction 4 B.v.P.I. sup^s rational integral

[or a quantic for 2 or more vars.]

5·1. *To find the quotient and the remainder when a polynomial $f(x)$ is divided by $x-a$.*

Remainder Theo.

Suppose $b_0x^{n-1}+b_1x^{n-2}+\ldots+b_{n-1}$ is the quotient and R, the remainder, so that we have the identity

$$\begin{aligned} f(x) &\equiv p_0x^n+p_1x^{n-1}+p_2x^{n-2}+p_3x^{n-3}+\ldots+p_{n-1}x+p_n \\ &\equiv (x-a)(b_0x^{n-1}+b_1x^{n-2}+b_2x^{n-3}+\ldots+b_{n-1})+R. \end{aligned}$$

Equating like powers of x on both sides of this identity, we have the series of equations:

$$\begin{aligned} p_0 &= b_0, && \therefore\ b_0 = p_0. \\ p_1 &= b_1 - ab_0, && \therefore\ b_1 = p_1 + ab_0 = p_1 + p_0a. \\ p_2 &= b_2 - ab_1, && \therefore\ b_2 = p_2 + ab_1 = p_2 + p_1a + p_0a^2. \\ p_3 &= b_3 - ab_2, && \therefore\ b_3 = p_3 + ab_2 = p_3 + p_2a + p_1a^2 + p_0a^3. \\ &\ldots\ldots\ldots\ldots && \ldots\ldots\ldots\ldots \\ p_r &= b_r - ab_{r-1}, && \therefore\ b_r = p_r + ab_{r-1} = p_r + p_{r-1}a + p_{r-2}a^2 + \ldots + p_0a^r. \\ &\ldots\ldots\ldots\ldots && \ldots\ldots\ldots\ldots \end{aligned}$$

and $p_n = R - ab_{n-1}$, $\therefore\ R = p_n + ab_{n-1}$

$$= p_n + p_{n-1}a + p_{n-2}a^2 + \ldots + p_0a^n = f(a).$$

We thus deduce the following scheme suggesting how $b_0, b_1, b_2, \ldots b_{n-1}$ can be calculated most expeditiously in practice.

	p_0	p_1	p_2	p_3	.....	p_{n-1}	p_n
	0	b_0a	b_1a	b_2a	.....	$b_{n-2}a$	$b_{n-1}a$
	b_0	b_1	b_2	b_3		b_{n-1}	R

The above process is known as **Synthetic Division**. It is of great use, particularly, in connexion with Horner's method of calculating the real roots of numerical equations (§ 6·9).

Example 1. Divide x^5-8x^3+1 by $x-3$.

Here $a=3$. Writing down the coefficients, zeros being written for absent terms, we have

1	0	−8	0	0	1
0	3	9	3	9	27
1	3	1	3	9	28.

Thus the quotient and the remainder are

$$x^4+3x^3+x^2+3x+9 \quad \text{and} \quad 28.$$

Example 2. Divide x^4+7x+1 by $x+2$.

Here $a=-2$; hence writing down the coefficients, we get

1	0	0	7	1
0	−2	4	−8	2
1	−2	4	−1	3.

Therefore the quotient and the remainder are respectively

$$x^3-2x^2+4x-1 \quad \text{and} \quad 3.$$

Examples. V. a.

Divide

1. x^5-4x^2+x+1 by x^2-1. 2. $5x^4+3x^2-9$ by $5x+1$.

3. x^5-x+8 by x^2-x-2; (4) $x^6-4x^4+3x^2+3$ by x^2+1.

5. Express n^3+1 in the form

$$A_0+A_1n+A_2n(n-1)+A_3n(n-1)(n-2).$$

6. Express n^4, $n^2(n+1)^2$; $n(n+1)(2n+1)$ similarly.

(For *Examples*, 5, and 6 read remarks on illustrative *example* 1, § 1·33)

5·2. *If for a polynomial $f(x)$, $f(a)=0$ then $x-a$ is a factor of $f(x)$.*

We have incidentally shewn above (§5·1) that if a polynomial $f(x)$ is divided by $x-a$, the remainder is $f(a)$. Hence if $f(a)=0$, the polynomial is exactly divisible by $x-a$. We can therefore write it in the form

$$f(x)=(x-a)F(x)$$

where $F(x)$ is a polynomial of the $n-1$th degree.

5·21. In the following theorem, we assume the truth of the fundamental theorem of algebra, that, for every polynomial $f(x)$ it is possible to find at least one value a of x, such that $f(a)=0$. Such a value is called a *root of the equation* $f(x)=0$.

An eqution of the nth degree has n and only n roots

Let the equation be

$$f(x)\equiv p_0x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_{n-1}x+p_n=0.$$

We know that at least one value a_1, of x exists, such that $f(a_1)=0$. Hence by §5·2, we have

$$f(x)=(x-a_1)f_1(x) \qquad (1)$$

where $f_1(x)$ is a polynomial of degree $n-1$. Applying the same argument to $f_1(x)$, a value a_2 of x exists such that $f_1(a_2)=0$. Hence by §5·2, we have

$$f_1(x)=(x-a_2)f_2(x) \qquad (2)$$

where $f_2(x)$ is a polynomial of degree $n-2$. From (1) and (2) we get

$$f(x)=(x-a_1)(x-a_2)f_2(x). \qquad (3)$$

Arguing repeatedly in this way, we have ultimately

$$f(x)\equiv p_0(x-a_1)(x-a_2)\ldots(x-a_n). \qquad (4)$$

It is clear from the right hand product of (4), that $a_1, a_2,\ldots a_n$ are n of the roots of $f(x)=0$. Now put a for x in the identity (4), a being supposed to be different from $a_1, a_2,\ldots a_n$. We then have

$$f(a)=p_0(a-a_1)(a-a_2)\ldots(a-a_n) \qquad (5)$$

It is evident that none of the factors $a-a_1, a-a_2,\ldots\ldots$ vanishes ; hence

$$f(a)\neq 0.$$

Hence the equation $f(x)=o$ has n and only n roots.

5·211. *If $f(x)=o$ has more than n distinct roots, then each of the coefficients of the equation is zero.*

For suppose $a_1, a_2, a_3,\ldots a_n, a_{n+1}$ are the roots of the eqnation; then by § 5·21, we have

$$f(x)=p_0(x-a_1)(x-a_2)\ldots(x-a_{n-1})(x-a_n). \qquad (1)$$

Since a_{n+1} is also a root, we have from (1)

$$f(a_{n+1})=p_0(a_{n+1}-a_1)(a_{n+1}-a_2)\ldots(a_{n+1}-a_n)=o \qquad (2)$$

Now none of the factors $a_{n+1}-a_1$, $a_{n+1}-a_2$, $a_{n+1}-a_n$ is zero,

$$\therefore\ p_0=0.$$

Applying the argument to the polynomial

$$p_1x^{n-1}+p_2x^{n-2}+p_3x^{n-3}+\ldots+p_n$$

which is of degree $n-1$, and which, by hypothesis, vanishes for more than $n-1$ values of x, we have $p_1=0$.

Arguing repeatedly in this way, we get

$$p_0=p_1=p_2=p_3=\ldots=p_{n-1}=p_n=0.$$

5·212. *If two polynomials $f(x)$ and $F(x)$ each of the nth degree, are equal for more than n distinct values of x, then the coefficients of like powers of x in the two polynomials are equal.*

It is clear that all those values of x for which the polynomials are equal, are roots of the equation $f(x)-F(x)=0$.

Since this equation is of the nth degree and has, by hypothesis, more than n roots, we deduce, from § 5·211, that all the coefficients in the polynomial $f(x)-F(x)$, must vanish. The required result is now quite obvious.

Example. Prove the identity

$$(x-b)(x-c)(b-c)+(x-c)(x-a)(c-a)+(x-a)(x-b)(a-b)+(b-c)(c-a)(a-b)=0.$$

It is easy to see that the left hand expression is of the second degree in x and that it vanishes for three values a, b, c of x. Hence all the coefficients must be zero. Therefore it must vanish for all values of x. It is easy to verify the fact directly.

5·213. We apply the above theorem (§ 5·211) to establish an important theorem attributed to Van der Monde.

It is usual to denote the product of r factors

$p(p-1)(p-2)(p-3)\ldots(p-r+1)$ by p_r.

e.g. $(-1)_r=(-1)^r r!$ and if p is a positive integer, $p_r \div r! = {}_pC_r$.

In this notation the theorem can be stated thus.

Binomial Theo. Prove that

$$(m+n)_r = m_r + {}_rC_1 m_{r-1}n_1 + {}_rC_2 m_{r-2}n_2 + {}_rC_3 m_{r-3}n_3 + \ldots + {}_rC_r n_r.$$

where m and n are any quantities.

If m and n are both positive integers, then by the binomial theorem, we have

$$(1+x)^m = 1+\frac{m_1}{1!}x+\frac{m_2}{2!}x^2+\ldots+\frac{m_r}{r!}x^r+\ldots+\frac{m_m}{m!}x^m$$

and $$(1+x)^n=1+\frac{n_1}{1!}x+\frac{n_2}{2!}x^2+\frac{n3}{3!}x^3+\ldots+\frac{n_r x^r}{r!}+\ldots+\frac{n_n}{n!}x^n.$$

Now the coefficient of the term in x^r in the product of the two right hand expansions is

$$\frac{m_r}{r!}+\frac{m_{r-1}\,n_1}{(r-1)!1!}+\frac{m_{r-2}.\,n_2}{(r-2)!2!}+\ldots\ldots+\frac{n_r}{r!}\ .$$

This quantity must be equal to the coefficient of the term in x^r in the expansion of $(1+x)^m\times(1+x)^n$ *i. e.* in $(1+x)^{m+n}$.

Hence it must be equal to ${}_{m+n}C_r$ or $(m+n)_r/r!$

We thus have

$$\frac{(m+n)_r}{r!}=\frac{m_r}{r!}+\frac{m_{r-1}.n_1}{(r-1)!1!}+\frac{m_{r-2}.\,n_2}{(r-2)!2!}+\ldots\ldots+\frac{n_r}{r!}\ .$$

Multiplying this relation by $r!$, we get

$$(m+n)_r=m_r+{}_rC_1 m_{r-1}\,n_1+{}_rC_2 m_{r-2}.n_2+\ldots+{}_rC_r n_r. \qquad (1)$$

Thus the theorem is true when m and n are positive integers.

Now the only restriction upon m and n implied in the above argument is that they should be positive integers not less than r. Hence the relation (1) which may be regarded as an equation in m of degree r, is satisfied by more than r values of m. Therefore by § 5.211, it must be an identity in m. Since m and n enter symmetrically into (1), it is also an identity in n.

Examples V. b.

Establish the following identities:

1. $$\frac{(x-b)(x-c)}{(a-b)(a-c)}+\frac{(x-c)(x-a)}{(b-c)(b-a)}+\frac{(x-a)(x-b)}{(c-a)(c-b)}=1.$$

2. $$a\frac{(x-b)(x-c)}{(a-b)(a-c)}+b\frac{(x-c)(x-a)}{(b-c)(b-a)}+c\frac{(x-a)(x-b)}{(c-a)(c-b)}=x.$$

3. $$a^2\frac{(x-b)(x-c)}{(a-b)(a-c)}+b^2\frac{(x-c)(x-a)}{(b-c)(b-a)}+c^2\frac{(x-a)(x-b)}{(c-a)(c-b)}=x^2.$$

4. $$a^3\frac{(x-b)(x-c)}{(a-b)(a-c)}+b^3\frac{(x-c)(x-a)}{(b-c)(b-a)}+c^3\frac{(x-a)(x-b)}{(c-a)(c-b)}$$
$$=x^3-(x-a)(x-b)(x-c).$$

5. From example (4) deduce that

$$\frac{x^3}{(x-a)(x-b)(x-c)}=1+\frac{a^3}{(x-a)(a-b)(a-c)}+\frac{b^3}{(x-b)(b-c)(b-a)}+\frac{c^3}{(x-c)(c-a)(c-b)}.$$

6. Prove that $(x+3)_3 - 3(x+3)_2 x + 3(x+3)_1 (x+1)_2 - (x+2)_3 = 3!$

$$\frac{1}{x} - \frac{3}{x+1} + \frac{3}{x+2} - \frac{1}{x+3} = \frac{3!}{x(x+1)(x+2)(x+3)};$$

and more generally

$$\frac{{}_nC_0}{x} - \frac{{}_nC_1}{x+1} + \frac{{}_nC_2}{x+2} - \frac{{}_nC_3}{x+3} + \ldots + \frac{(-1)^r {}_nC_r}{x+r} + \ldots + \frac{(-1)^n {}_nC_n}{x+n} = \frac{n!}{x(x+1)(x+2)\ldots(x+n)}.$$

5·22. If $a_1, a_2, a_3, \ldots a_n$ are the roots of

$$f(x) = p_0 x^n + p_1 x^{n-1} + p_2 x^{n-2} + p_3 x^{n-3} + \ldots\ldots + p_{n-1} x + p_n = 0$$

then $\Sigma_1 = (-1)^1 p_1/p_0$, $\Sigma_2 = (-1)^2 p_2/p_0$, $\Sigma_3 = (-1)^3 p_3/p_0, \ldots,$

$$\Sigma_r = (-1)^r p_r/p_0 \ldots .$$

where Σ_r *stands for the sum of products of* $a_1, a_2, \ldots, a_n$ *taken* r *at a time, i.e.* (1^r).

We have by §5·21,

$$p_0 x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_n = p_0 (x-a_1)(x-a_2)(x-a_3)\ldots(x-a_n).$$

Distributing the right hand product, we have

$$p_0 x^n + p_1 x^{n-1} + \ldots + p_n = p_0 [x^n - \Sigma_1 x^{n-1} + \Sigma_2 x^{n-2} + \ldots (-1)^r \Sigma_r x^{n-r} - \ldots]$$

Equating the coefficients of like powers on both sides of this identity, we have

$$p_0 = p_0,\ p_1 = (-1)^1 p_0 \Sigma_1,\ p_2 = (-1)^2 p_0 \Sigma_2,\ p_r = (-1)^r p_0 \Sigma_r, \ldots\ldots$$

from which the required results are immediately derived.

In part:. (i) If a_1, a_2 are the roots of $p_0 x^2 + p_1 x + p_2 = 0$, then

$$a_1 + a_2 = -p_1/p_0, \text{ and } a_1 a_2 = p_2/p_0.$$

(ii) Similarly if a_1, a_2, a_3 are the roots of the cubic equation,

$$p_0 x^3 + p_1 x^2 + p_2 x + p_3 = 0,$$

then

$$a_1 + a_2 + a_3 = -p_1/p_0,$$

$$a_2 a_3 + a_3 a_1 + a_1 a_2 = p_2/p_0,$$

and

$$a_1 a_2 a_3 = -p_3/p_0.$$

5·3. *Prove that*

$$f(x+h) = f(x) + \frac{h}{1!} f'(x) + \frac{h^2}{2!} f''(x) + \ldots\ldots + \frac{h^r}{r!} f^{(r)}(x) + \ldots + \frac{h^n}{n!} f^{(n)}(x).$$

where we use the usual notation of the calculus.

Now $f(x+h) = p_0(x+h)^n + p_1(x+h)^{n-1} + p_2(x+h)^{n-2} + \ldots$
$+ p_r(x+h)^r + \ldots + p_n.$

Applying the binomial theorem for a positive integral exponent to each of $(x+h)^n$, $(x+h)^{n-1}$,..., we have

$p_0(x+h)^n = p_0(x^n + {}_nC_1x^{n-1}h + {}_nC_2x^{n-2}h^2 + {}_nC_3x^{n-3}h^3 + \ldots + {}_nC_nh^n)$,

$p_1(x+h)^{n-1} = p_1(x^{n-1} + {}_{n-1}C_1x^{n-2}h + {}_{n-1}C_2x^{n-3}h^2 + {}_{n-1}C_3x^{n-4}h^3 + \ldots$
$+ {}_{n-1}C_{n-1}h^{n-1})$,

$p_2(x+h)^{n-2} = p_2(x^{n-2} + {}_{n-2}C_1x^{n-3}h + {}_{n-2}C_2x^{n-4}h^2 + \ldots$
$+ {}_{n-2}C_rx^{n-r-2}h^r + \ldots)$,

.....

$p_{n-1}(x+h) = p_{n-1}x + p_{n-1}h$,

and $p_n = p_n 1$.

Adding up these relations and arranging according to powers of h, we have the required result. It is only a simple particular case of Taylor's Theorem in calculus. (See Lamb's *Infinitesimal calculus*, § 193.)

(i) *e. g.* If $f(x) = p_0x^2 + p_1x + p_2$,

then $f(x+h) = p_0x^2 + p_1x + p_2 + h(2p_0x + p_1) + \frac{h^2}{2!}(2p_0)$.

(ii) Again if $f(x) = p_0x^3 + p_1x^2 + p_2x + p_3$, then

$f(x+h) = p_0x^3 + p_1x^2 + p_2x + p_3 + h(3p_0x^2 + 2p_1x + p_2)$
$+ h^2(3.2p_0x + 2p_1)/2! + h^3 3.2.1 p/3!$.

5·31. *Prove that* $\frac{1}{f(x)} \sum \frac{1}{x-a_r} = \frac{f'(x)}{f(x)}$

$f'(x) = np_0x^{n-1} + (n-1)p_1x^{n-2} + (n-2)p_2x^{n-3} + \ldots + p_{n-1}$.

$= f(x)\left\{\frac{1}{x-a_1} + \frac{1}{x-a_2} + \frac{1}{x-a_3} + \ldots + \frac{1}{x-a_n}\right\}$

where $a_1, a_2, a_3, \ldots a_n$ *are the roots of* $f(x) = 0$.

Now by § 5·21, we have the identity

Rather, assume $f(x) = p_0(x-a_1)(x-a_2)(x-a_3)\ldots(x-a_n)$.

Writing $x+h$ for x in this identity, we get

$f(x+h) = p_0(x+h-a_1)(x+h-a_2)(x+h-a_3)\ldots(x+h-a_n)$,

$= p_0[(x-a_1)+h][(x-a_2)+h][(x-a_3)+h]\ldots\ [(x-a_n)+h]$

$= p_0(P_0 + P_{n-1}h + P_{n-2}h^2 + P_{n-3}h^3 + \ldots + h^n)$

where P_r stands for the sum of products of $x-a_1$, $x-a_2$,..., $x-a_n$ taken r at a time. Now from § 5·3,

By Taylor's Theo. $f(x+h)=f(x)+\frac{h}{1!}f'(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3}f'''(x)+\ldots\ldots$

Thus we have the identity.

$$f(x)+\frac{h}{1!}f'(x)+\frac{h^2}{2!}f''(x)+\ldots=p_0(P_1+P_{n-1}h+\ldots).$$

Equating the coefficients of h on both sides, we get

What abt. coffs of h^2? $f'(x)=p_0P_{n-1}$

$$=p_0(x-a_1)(x-a_2)(x-a_3)\ldots(x-a_n)\left\{\frac{1}{x-a_1}+\frac{1}{x-a_2}+\ldots\right\}$$

$$=f(x)\left\{\frac{1}{x-a_1}+\frac{1}{x-a_2}+\frac{1}{x-a_3}+\ldots+\frac{1}{x-a_n}\right\}.$$

Now $f'(x)$ is also equal to $np_0x^{n-1}+(n-1)p_1x^{n-2}+\ldots$Hence etc.

N.B. The above result may also be obtained easily by applying the rule for differentiating a product. i.e., by diff.

Examples V. c.

1. Prove that (when x is a variable)
(the poly. in a def. x)

$$f(x+h)=f(h)+\frac{x}{1!}f'(h)+\frac{x^2}{2!}f''(h)+\frac{x^3}{3!}f'''(h)+\ldots$$

2. If $a_1, a_2, a_3, \ldots, a_n$ are the n distinct roots of $f(x)=0$, prove by diff. $f(a)$ & subj. a for x, etc. that

$$f'(a_1)=p_0(a_1-a_2)(a_1-a_3)(a_1-a_4)\ldots(a_1-a_n)$$
$$f'(a_2)=p_0(a_2-a_1)(a_2-a_3)(a_2-a_4)\ldots(a_2-a_n)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Repeated roots.

Smith 573. 3. Prove that if $f(x)\equiv(x-a)^rF(x)$, then

$$f'(x)=(x-a)^{r-1}[rF(x)+(x-a)F'(x)].$$

4. If $f(x)\equiv(x-a)^p(x-b)^qF(x)$, prove that

$$f'(x)=(x-a)^{p-1}(x-b)^{q-1}[(x-a)(x-b)F'(x)+[(p+q)x-pb-qa]F(x)].$$

5. Prove that

$$f(x)=f(a)+\frac{(x-a)}{1!}f'(a)+\frac{(x-a)^2}{2!}f''(a)+\frac{(x-a)^3}{3!}f'''(a)+\ldots$$

Sums of powers of roots. Rather confused. In Smith 594 S_r = Σ(products of r roots)

5·4. *To express* S_r *where* $S_r=a_1^r+a_2^r+\ldots\ldots+a_n^r$ $(r<n)$ *in terms of the coefficients of the equation* $f(x)=0$. ($r>n$ (108) obvious Smith 557)

Smith (556) 591 99 101 Grace to prods. in Chrys. II 212 27

We have by §5·1,

N.B. Smith states (5·55) with any n equations. & then (5·56) considers them as roots.

By division

$$f(x)/(x-a_1) = p_0x^{n-1} + (p_1+a_1p_0)x^{n-2} + (p_2+p_1a_1+p_0a_1^2)x^{n-3} + \dots$$
$$\dots + (p_r+p_{r-1}a_1+p_{r-2}a_1^2+p_{r-3}a_1^3+\dots+p_0a_1^r)x^{n-r-1}+\dots$$
$$f(x)/(x-a_2) = p_0x^{n-1} + (p_1+p_0a_2)x^{n-2} + (p_2+p_1a_2+p_0a_2^2)x^{n-3}+\dots$$
$$+ (p_r+p_{r-1}a_2+p_{r-2}a_2^2+p_{r-3}a_2^3+\dots+p_0a_2^r)x^{n-r-1}+\dots$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
and $f(x)/(x-a_n) = p_0x^{n-1} + (p_1+a_np_0)x^{n-2} + (p_2+p_1a_n+p_0a_n^2)\,x^{n-3}$
$$+\dots+(p_r+p_{r-1}a_n+p_{r-2}a_n^2+p_{r-3}a_n^3+\dots+p_0a_n^r)x^{n-r-1}+\dots$$

Adding up all these identities, we have

$$f(x)\left\{\frac{1}{x-a_1}+\frac{1}{x-a_2}+\frac{1}{x-a_3}+\dots+\frac{1}{x-a_n}\right\}$$
$$= np_0x^{n-1} + (np_1+p_0S_1)x^{n-2} + (np_2+p_1S_1+p_0S_2)x^{n-3}+\dots$$
$$+ (np_r+p_{r-1}S_1+p_{r-2}S_2+\dots+p_0S_r)x^{n-r-1}+\dots \qquad (1)$$

Now by § 5·31, the left hand expression is equal to

By logarithmic diff.

$$np_0x^{n-1}+(n-1)p_1x^{n-2}+(n-2)p_2x^{n-3}+\dots+(n-r)p_rx^{n-r-1}+\dots \qquad (2)$$

Hence, we have the identity

$$np_0x^{n-1}+(n-1)p_1x^{n-2}+\dots+(n-r)p_rx^{n-r-1}+\dots$$
$$= np_0x^{n-1} + (np_1+p_0S_1)x^{n-2} + (np_2+p_1S_1+p_0S_2)x^{n-3}+\dots$$
$$+ (np_r+p_{r-1}S_1+p_{r-2}S_2+\dots+p_0S_r)x^{n-r-1}+\dots \qquad (3)$$

Equating the coefficients of like powers of x on both sides of the identity (3), we have the series of formulae attributed to Newton:-

(No double accent: Ball 231)

Brit. I 165-171

S_r for $r < n$

$$p_0S_1 + 1p_1 = 0,$$
$$p_0S_2 + p_1S_1 + 2p_2 = 0, \quad \text{i.e. } \Sigma x^2 = (\Sigma x)^2 - 2\Sigma x_1x_2$$
$$p_0S_3 + p_1S_2 + p_2S_1 + 3p_3 = 0, \quad \text{i.e. } \Sigma x^3 = (\Sigma x)(\Sigma x^2) - (\Sigma x_1x_2)(\Sigma x) + 3\Sigma x_1x_2x_3$$
$$p_0S_4 + p_1S_3 + p_2S_2 + p_3S_1 + 4p_4 = 0, \quad \text{i.e. } \Sigma x^4 = \Sigma x\,\Sigma x^3 - (\Sigma x_1x_2)\Sigma x^2 + \dots$$
$$\text{i.e. } \sum_{r=1}^{n-1}(\text{homog. sum of products of roots})(\Sigma(\text{root})^r) = r\,\Sigma(\text{homog. sum of products})$$
[Any connection with Euler's diff$^{\text{l}}$ of homog. func$^{\text{n}}$?]
$$\rightarrow\quad p_0S_r + p_1S_{r-1} + p_2S_{r-2} + \dots + p_4S_{r-4} + \dots + rp_r = 0, \quad = \sum_{s=0}^{r} p_s S_{r-s}$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
and $p_0S_{n-1} + p_1S_{n-2} + p_2S_{n-3} + \dots + (n-1)p_{n-1} = 0. \quad = \sum_{s=0}^{n-1} p_s S_{n-1-s}$

S_r for $r < n$

By means of these formulae, we can express p's in terms of S's and S's, in terms of p's; for instance, by eliminating p_1, p_2 from the first three (§ 2·64, illustrative *example* 2), we have

(i) Applications

$$\begin{vmatrix} p_0S_1 & 1 & 0 \\ p_0S_2 & S_1 & 2 \\ p_0S_3+3p_3 & S_2 & S_1 \end{vmatrix} = 0, \text{ or } p_0\begin{vmatrix} S_1 & 1 & 0 \\ S_2 & S_1 & 2 \\ S_3 & S_2 & S_1 \end{vmatrix} + 3\,p_3\begin{vmatrix} 1 & 0 \\ S_1 & 2 \end{vmatrix} = 0.$$

Similarly eliminating S_1, S_2 from the same equations,

(ii) Corr. formulae for Σ powers of reciprocals of roots. (See Smith 5·56)

$$p_0S_1' + p_{n-1} = 0, \dots\dots \sum_{s=0} p_{n-s}\,S'_{n-s} = 0$$
$$\text{for } r > n \quad \sum_{s=0} \dots = 0$$

$$\begin{vmatrix} p_0 & 0 & p_1 \\ p_1 & p_0 & 2p_2 \\ p_2 & p_1 & 3p_3+p_0S_3 \end{vmatrix} =0, \quad \therefore \quad (-p_0)^3S_3=\begin{vmatrix} p_1 & p_0 & 0 \\ 2p_2 & p_1 & p_0 \\ 3p_3 & p_2 & p_1 \end{vmatrix}.$$

To get S_{n+r}, we put $a_1, a_2, \ldots a_n$, for x successively in

$$x^r f(x)=p_0x^{n+r}+p_1x^{n+r-1}+p_2x^{n+r-2}+\ldots+p_nx^r=0$$

and then add them. We have thus

$$p_0S_{n+r}+p_1S_{n+r-1}+p_2S_{n+r-2}+\ldots+p_nS_r=0.$$

By putting $r=0$ we get an equation expressing S_n in terms of the coefficients, and $S_{n-1}, S_{n-2}, \ldots S_1$, which we know, can be expressed also in terms of the coefficients. In this way, if it is possible to express S_m in terms of the coefficients for all values of m.

5·41. The following method of obtaining the above formulae is perhaps more direct and expeditious. Putting $xy=1$ in the identity (§ 5·31),

$$\frac{f'(x)}{f(x)}=\frac{1}{x-a_1}+\frac{1}{x-a_2}+\frac{1}{x-a_3}+\ldots+\frac{1}{x-a_n}$$

we have

$$\frac{np_0+(n-1)p_1y+(n-2)p_2y^2+\ldots+p_{n-1}y^{n-1}}{p_0+p_1y+p_2y^2+\ldots+p_ny^n}$$

$$=\frac{1}{1-a_1y}+\frac{1}{1-a_2y}+\frac{1}{1-a_3y}+\ldots+\frac{1}{1-a_ny}.$$

Now $1/(1-a_1y)=1+a_1y+a_1^2y^2+\ldots+a_1^my^m+a_1^{m+1}y^{m+1}/(1-a_1y)$;

$1/(1-a_2y)=1+a_2y+a_2^2y^2+\ldots+a_2^my^m+a_2^{m+1}y^{m+1}/(1-a_2y)$;

.....

and $1/(1-a_ny)=1+a_ny+a_n^2y^2+\ldots+a_n^my^m+a_n^{m+1}y^{m+1}/(1-a_ny)$

Adding up all these identities, we have

$$\frac{np_0+(n-1)p_1y+(n-2)p_2y^2+\ldots+p_{n-1}y^{n-1}}{p_0+p_1y+p_2y^2+\ldots+p_ny^n}$$

$$=n+S_1y+S_2y^2+S_3y^3+\ldots+S_ry^r+\ldots$$

$$+S_my^m+y^{m+1}F(y)/y^nf(1/y) \quad (1)$$

where m is an integer as large as we please and $F(y)$ is a polynomial in y of lower degree than n.

Hence we derive the rule:

To obtain S_r for any equation $f(x)=0$, divide $x^{n-1}f'(1/x)$ by $x^nf(1/x)$ by the Process of Ascending Continued Division, and equate S_r to the coefficient of the term in x^r.

It is interesting to see how the identity (1) provides a very simple method of obtaining the whole set of *Newton's formulae*. For cléaring of fractions, we get

$$np_0+(n-1)p_1y+(n-2)p_2y^2+\ldots+p_{n-1}y^{n-1}$$
$$=(n+S_1y+S_2y^2+\ldots+S_my^m)(p_0+p_1y+\ldots+p_ny^n)+y^{m+1}F(y).$$

Now equating coeffi:ients of like powers of y on both sides of this identity we get the required resnlts.

Example 1. Calculate S_2, S_3, S_4, S_5 for the cubic, $x^3+px-r=0$ and prove that $S_5/5=(S_3/3)(S_2/2)$.

We have the identity, $3+qy^2=(1+qy^2-ry^3)(3+S_1y+S_2y^2+\ldots+S_5y^5)+y^6F(y)$ where $F(y)$ is a polynomial. Equating coefficients of like powers of y, we get.

$$0=S_1;\ q=S_2+3q;\ 0=S_3+qS_1-3r,\ 0=S_4+qS_2-rS_1$$

and

$$0=S_5+qS_3-rS_2.$$

From these, we have

$S_2=-2q$, $S_3=3r$, $S_4=2q^2$ and $S_5=rS_2-qS_3=5S_2.S_3/2.3.=\frac{5}{6}(-2q)(3r)$ as on 104

$$\therefore\quad S_5/5=(S_2/2)(S_3/3).$$

Example 2. If $x+y+z=0$ prove that $2(x^7+y^7+z^7)=7xyz(x^4+y^4+z^4)$. [*B. U. 1920 Problems*]

Denoting $yz+zx+xy$ and xyz by q and r, the equation whose roots are x, y, z, is given by $t^3+qt-r=0$

Using Newton's formulae, we have (why not the reverse order?)

$S_7+qS_5-rS_4=0$; $S_5+qS_3-rS_2=0$, $S_4+qS_2=0$, $S_3-3r=0$, $S_2=-2q$.

From these, we have

$$S_2=-2q,\ S_3=3r;\ S_4=2q^2,\ S_5=-5qr \text{ and } S_7=7q^2r.$$

Now $2(x^7+y^7+z^7)=2S_7=14q^2r$ and $7xyz(x^4+y^4+z^4)=7r.2q^2$ and hence

$$2(x^7+y^7+z^7)=7xyz(x^4+y^4+z^4).$$

Examples. V. d.

1. Solve the equations $x+y+z=0$, $x^2+y^2+z^2=14$, and $x^3+y^3+z^3=-18$.
2. If $S_1=S_2=0$, $S_3=-6$ and $S_4=18$, prove that $S_6=12$ and $S_7=-63$ where $S_r=a_1^r+a_2^r+a_3^r+a_4^r$.
3. If $a_1+a_2+a_3+a_4=0$, prove that

$$(a_1^5+a_2^5+a_3^5+a_4^5)\div 5=\tfrac{1}{2}(a_1^2+a_2^2+a_3^2+a_4^2)\tfrac{1}{3}(a_1^3+a_2^3+a_3^3+a_4^3)$$

4. If $a+b+c+\ldots+k+l=0$, prove that
$a^3+b^3+c^3+\ldots+k^3+l^3=3(abc+abd+\ldots+abl+\ldots)$.

5. If $S_1=S_{-1}=0$, then $S_{-2r-1}=S_{2r+1}=0$, where
$$S_m=a_1^m+a_2^m+a_3^m+a_4^m$$
for all positive and negative integral values of m.

6. If $n>2$, prove, for the epuation $x^n+px+q=0$,
$S_1=S_2=S_3=\ldots=S_{n-2}=0$ and $\frac{S_{2n-1}}{2n-1}=\frac{S_{n-1}}{n-1}\frac{S_n}{n}$.

7. If for an equation $x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_n=0$,
$S_1=S_2=S_3=\quad=S_r=0$ $(r<n)$, then show that
$$p_1=p_2=p_3=\ldots=p_r=0\ .$$

8. Prove that a, b, c are the roots of

$$\begin{vmatrix} x^3 & x^2 & x & 1 \\ S_1 & 1 & 0 & 0 \\ S_2 & S_1 & 2 & 0 \\ S_3 & S_2 & S_1 & 3 \end{vmatrix}=0, \text{ and also of } \begin{vmatrix} x^3 & x^2 & x & 1 \\ S_p & S_{p-1} & S_{p-2} & S_{p-3} \\ S_q & S_{q-1} & S_{q-2} & S_{q-3} \\ S_r & S_{r-1} & S_{r-2} & S_{r-3} \end{vmatrix}=0,$$

where $S_m=a^m+b^m+c^m$, p, q, r being three unequal integers each greater than 3.

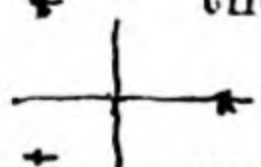
5·411. *Cube roots of unity*–These roots clearly satisfy the equation $x^3-1=0$. Now $x^3-1=(x-1)(x^2+x+1)$.

Hence they are determined by $x-1=0$ and $x^2+x+1=0$.

Now the roots of $x^2+x+1=0$ are $(-1\pm i\sqrt{3})\div 2$. Hence the cube roots of unity are 1, $(-1+i\sqrt{3})\div 2$, , $(-1-i\sqrt{3})\div 2$.

5.412 Putting the complex roots into their normal form, the roots are 1, $\cos\frac{2}{3}\pi+i\sin\frac{2}{3}\pi$; $\cos\frac{4}{3}\pi+i\sin\frac{4}{3}\pi$ or $\cos\frac{2}{3}\pi-i\sin\frac{2}{3}\pi$.

If we denote $\cos\frac{2}{3}\pi+i\sin\frac{2}{3}\pi$ by w, the other complex root is then either w^2 or w^{-1}.



It is easy to prove that $1+w+w^2=1+w^2+w^4=0$; more generally, the sum of the mth powers of the cube roots of unity is either 3 or 0 according as m is or is not a multiple of 3. (Apply §5·41 or prove directly).

In the same notation, the following identities are easily verified.

1. $x^2+x+1=(x-w)(x-w^2)$;

2. $x^3-y^3=(x-y)(x-yw)(x-yw^2)$;

3 $x^3+y^3=(x+y)(x+yw^2)(x+yw)$

4. $x^3+y^3+z^3-3xyz=(x+y+z)(x+yw+zw^2)(x+yw^2+zw)$

5·5. Symmetric function of roots.

If, after replacing a by b, and b by a, a function of a and b does not change in any respect, it is said to be symmetric with respect to a and b. *e. g.* ab; $a+b$; $(a-b)^2$, and $\sqrt{a}+\sqrt{b}+\sqrt{(ab)}$. More generally if a function of n quantities $a_1, a_2, \ldots a_n$, is symmetric with respect to every two of them, it is said to be symmetric with respect to all of them; $a+b+c$; abc, $a+b+c+a^2+b^2+c^2$; and $(a+wb+w^2c)^3+(a+w^2b+wc)^3$ where w is a complex cube root of unity, are instances of functions symmetric with respect to a, b, c. Symmetric functions of roots play a very important part in the theory of equations. The following property regarding them is important.

5·51. *A rational symmetric function of the roots of an equation $f(x)\equiv 0$ can be always evaluated rationally in terms of the coefficients of the equation.*

It is easy to see that the numerator and the denominator of a rational symmetric function, are integral symmetric functions involving expressions like $\Sigma a_1^p a_2^q$, $\Sigma a_1^p a_2^q a_3^r$. It is therefore sufficient to prove the theorem in the case of expressions like $\Sigma a_1^p a_2^q$ and $\Sigma a_1^p a_2^q a_3^r$, etc.

Consider the product

$$(a_1^p+a_2^p+a_3^p+\ldots+a_n^p)\times(a_1^q+a_2^q+a_3^q+\ldots+a_n^q)$$

where p and q are two unequal positive integers. The only types of terms that occur in its distribution are a_1^{p+q} and $a_1^p a_2^q$. It is further clear, that each of these terms occurs only once; hence we have

$$S_pS_q=\Sigma a_1^{p+q}+\Sigma a_1^p a_2^q;$$

$$\therefore \Sigma a_1^p a_2^q=S_pS_q-S_{p+q}. \qquad (1)$$

Now S_p, S_q, S_{p+q} can be expressed rationally in terms of the coefficients (§ 5·4); hence from (1), we see that an expression of the form $\Sigma a_1^p a_2^q$ can be expressed rationally in terms of the coefficients.

Cor. If $p=q$, pairs of terms like $a_1^p a_2^q$ and $a_1^q a_2^p$ become identical and the identity (1) reduces to the form

$$2\Sigma a_1^p a_2^p = (S_p)^2 - S_{2p}. \qquad (1').$$

Thus in this case also the theorem is true.

Now consider the product

$$S_p S_q S_r = (a_1^p + a_2^p + \ldots + a_n^p)(a_1^q + a_2^q + \ldots + a_n^q) \times (a_1^r + a_2^r + \ldots + a_n^r)$$

where, as before, p, q, r are distinct positive integers

The distinct types of terms that occur in its distribution are $a_1^p a_2^q a_3^r$, $a_1^{p+q} a_2^r$, $a_1^{p+r} a_2^q$, $a_1^{q+r} a_2^p$ and a_1^{p+q+r}.

Further each of these terms occurs only once; hence

$$S_p S_q S_r = \Sigma a_1^p a_2^q a_3^r + \Sigma a_1^{p+q} a_2^r + \Sigma a_1^{q+r} a_2^p + \Sigma a_1^{r+p} a_2^q + \Sigma a_1^{p+q+r}. \qquad (2)$$

Now by (1) and § 5·4. all expressions in (2) other than $\Sigma a_1^p a_2^q a_3^r$ can be evaluated rationally in terms of the coefficients of the equation. Hence the theorem holds in the case of expressions like $\Sigma a_1^p a_2^q a_3^r$.

Cor. 1 When $p=q=r$, six terms like

$a_1^p a_2^q a_3^r$, $a_1^p a_3^q a_2^r$, $a_2^p a_1^q a_3^r$, $a_2^p a_3^q a_1^r$, $a_3^p a_1^q a_2^r$, and $a_3^p a_2^q a_1^r$

corresponding to the 3! or 6 permutations of the suffixes 1, 2, 3, become identical in $\Sigma a_1^p a_2^q a_3^r$. Hence in this case (2) reduces to the form

$$(S_p)^3 = 6\Sigma a_1^p a_2^p a_3^p + 3\Sigma a_1^{2p} a_2^p + S_{3p}.$$

The theorem is thus true of this case also.

Cor. 2. Further it is not difficult to see, that we arrive at the same conclusion, when only two of the integers p, q, r become equal.

S. Smith. We may thus shew successively that symmetric functions of types $\Sigma a_1^p a_2^q a_3^r a_4^s$ etc. obey the theorem.

The method of evaluating symmetric functions of roots suggested in the last article, involves much labour; great economy of labour can however be secured by the use of intelligent devices that are often suggested by the form of the function.

Examples 1. If a, b, c, d be the roots of the equation

$$x^4 + px^3 + qx^2 + rx + s = 0$$

find (i) $\Sigma a^2 b^2$ (ii) $\Sigma a^3 b$. [*B. U. 1926 Hons.* I.]

(i) Now $q^2=(\Sigma ab)^2=\Sigma a^2b^2+2\Sigma a^2bc+6abcd.$

$$\therefore q^2=\Sigma a^2b^2+2\Sigma a^2bc+6s. \quad (1)$$

Further $\Sigma a^2bc=\Sigma abc\Sigma a-4abcd=pr-4s.$ (2)

Hence from (1) and (2), we get

$$\underline{\Sigma a^2b^2}=q^2-6s-2(pr-4s)=q^2-2pr+2s.$$

(ii) By (1) of § 5·51, $\Sigma a^3b=S_3S_1-S_4.$

Now from Newton's formulae (§ 5·4), we have

$S_1=-p;\ S_2=p^2-2q;\ S_3=-p(p^2-2q)+qp-3r$ and

$$S_4=p^2(p^2-2q)-qp^2+3pr-q(p^2-2q)+pr-4s$$
$$=p^4-4p^2q+4pr-4s+2q^2.$$

Hence $\underline{\Sigma a^3b}=p^2(p^2-2q)-p^2q+3pr-p^4+4p^2q-4pr+4s-2q^2$
$=p^2q-pr+4s-2q^2.$

The second part may also be obtained thus:—

Consider the product $\Sigma ab\Sigma a^2$. The types of terms that occur in the expansion of the product are a^3b, a^2bc. A little consideration shows that each of them occurs only once; hence

$$\Sigma ab\Sigma a^2=\Sigma a^3b+\Sigma a^2bc.$$

In the same way $\Sigma a^2bc=\Sigma abc\Sigma a-4abcd.$

From these, we have

$$\underline{\Sigma a^3b}=\Sigma ab\Sigma a^2-\Sigma abc\Sigma a+4abcd=q(p^2-2q)-pr+4s.$$

It is thus seen that the second method is more direct and less laborious. It requires, however, greater skill and more accurate knowledge of the theory of combinations on the part of the student.

Example 2. If $a_1, a_2, a_3, \ldots a_n$, are the roots of the equation

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_n=0,$$

find the value of the symmetric function

$$\Sigma\frac{a_1^2+a_2^2}{a_1a_2}.$$ [*B. U.* 1925 *Hons. I*].

Now $$\Sigma\frac{a_1^2+a_2^2}{a_1a_2}=\Sigma\left\{\frac{a_1}{a_2}+\frac{a_2}{a_1}\right\}.$$

Hence the sum of the terms which contain a_1 in the numerator is

$$a_1[a_2^{-1}+a_3^{-1}+a_4^{-1}+\ldots+a_n^{-1}]=a_1\Sigma a_1^{-1}-1.$$

Similarly the sum of terms containing a_2 in the numerator is

$$a_2\Sigma a_1^{-1}-1.$$

Hence $\Sigma \dfrac{a_1^2+a_2^2}{a_1a_2} = a_1\Sigma\dfrac{1}{a_1}+a_2\Sigma\dfrac{1}{a_1}+\ldots+a_n\Sigma\dfrac{1}{a_1}-n$

$$= \Sigma a_1 \Sigma\frac{1}{a_1}-n \ .$$

Now $\Sigma\dfrac{1}{a_1}$ = (The sum of products taken $n-1$ at a time)$/a_1a_2\ldots a_n$

$$= -p_{n-1}/p_n$$

Hence $$\Sigma\frac{a_1^2+a_2^2}{a_1a_2} = \frac{p_1p_{n-1}}{p_n}-n \ .$$

Example 3. Evaluate the symmetric function $(t_2+t_3)(t_3+t_1)(t_1+t_2)$ for the equation $x^3+ax^2+bx+c=0$.

Since $t_1+t_2+t_3=-a$

$$(t_2+t_3)(t_3+t_1)(t_1+t_2) = (-a-t_1)(-a-t_2)(-a-t_3). \qquad (1)$$

Now we have the identity

$$x^3+ax^2+bx+c \equiv (x-t_1)(x-t_2)(x-t_3).$$

Putting $-a$ for x in this identity, we have

$$-a^3+a^3-ab+c = (-a-t_1)(-a-t_2)(-a-t_3). \qquad (2)$$

From (1), (2), we get $(t_2+t_3)(t_3+t_1)(t_1+t_2) = c-ab$.

Example 4. Calculate $(t_3-t_1)^2(t_1-t_3)^2(t_2-t_3)^2$ for the cubic

$$x^3+3bx+c=0.$$

Since $t_1+t_2+t_3=0$, and $t_1\,t_2\,t_3=-c$,

$$(t_2-t_3)^2 = (t_2+t_3)^2-4t_2t_3 = t_1^2-4t_2t_3 = (t_1^3+4c)/t_1.$$

Now $t_1^3+3bt_1+c=0$; hence $t_1^3+4c = 3c-3bt_1$.

$$\therefore\ (t_2-t_3)^2(t_3-t_1)^2(t_1-t_2)^2 = \frac{27b^3}{t_1t_2t_3}\left\{\frac{c}{b}-t_1\right\}\left\{\frac{c}{b}-t_2\right\}\left\{\frac{c}{b}-t_3\right\}.$$

Now $$x^3+3bx+c \equiv (x-t_1)(x-t_2)(x-t_3).$$

Putting $x=c/b$, we have $c^3/b^3+4c = (c/b-t_1)(c/b-t_2)(c/b-t_3)$

Hence $(t_2-t_3)^2(t_3-t_1)^2(t_1-t_2)^2 = -27(c^2+4b^3)$.

Example 5. If a, b, c, d are the roots of the equation

$$x^4+px^3+qx^2+rx+s=0$$

prove that

$$\begin{vmatrix} 1 & bcd & bcd & bcd \\ dca & 1 & dca & dca \\ abd & abd & 1 & abd \\ abc & abc & abc & 1 \end{vmatrix} = 1-qs-2ps^2-3s^3.$$

[*B. U.* 1922 *Problems.*]

Denote the determinant by Δ and multiply its rows by a, b, c, d, it then transforms into

$$s\,\Delta = \begin{vmatrix} a & s & s & s \\ s & b & s & s \\ s & s & c & s \\ s & s & s & d \end{vmatrix}$$

s being substituted for $abcd$.

Now, $$s\,\Delta = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ o & a & s & s & s \\ o & s & b & s & s \\ o & s & s & c & s \\ o & s & s & s & d \end{vmatrix}.$$

Now multiplying the first row by s and then subtracting it from each of the rows, we have

$$s\,\Delta = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ -s & a-s & 0 & 0 & 0 \\ -s & 0 & b-s & 0 & 0 \\ -s & 0 & 0 & c-s & 0 \\ -s & 0 & 0 & 0 & a-s \end{vmatrix}.$$

Expanding the determinant in terms of the elements of the first column, we have

$$\begin{aligned} s.\,\Delta &= (a-s)(b-s)(c-s)(d-s) + (b-s)(c-s)(d-s)s \\ &\quad + (c-s)(d-s)(a-s)s + (d-s)(a-s)(b-s)s + (a-s)(b-s)(c-s)s \\ &= f(s) - sf'(s), \qquad \text{(see § 5·31)} \end{aligned}$$

where $f(x) = x^4 + px^3 + qx^2 + rx + s$.

Hence $\Delta = 1 - qs - 2ps^2 - 3s^3$.

Examples. V. e.

If a, b, c are the roots of the cubic, $f(x) \equiv x^3 + px^2 + qx + r = 0$, calculate the following symmetric functions:–

1. $\Sigma(b+c)^2$. 2. $\Sigma(b-c)^2$. 3. $\Sigma[(lb+mc)^2 + (lc+mb)^2]$.

4. Σb^3c^2. 5. $\Sigma(b^{-1}+c^{-1})-1$. 6. $\Sigma(b+c^{-1})(c+b^{-1})$.

7. $\Pi(2a-b-c)$. 8. $\Pi(b^{-1}+c^{-1})$. 9. $\Pi(a-bc)$.

10. $\Pi(b+c+bc)$. 11. $\Pi(b+c^{-1})(c+b^{-1})$. 12. $\Pi(a^2-bc)$.

13. $\Sigma(a^2-bc)^{-1}$. 14. $\Sigma(b+c)/(bc-a^2)$.

15. $(a+bw+cw^2)^3 + (a+bw^2+cw)^3$ where w is a complex cube root of unity.

If a, b, c, d are the roots of the equation

$$f(x) \equiv x^4+px^3+qx^2+rx+s=0,$$

evaluate the following symmetric functions.

16. $(b+c+d)(c+d+a)(d+a+b)(a+b+c)$.

17. $(b+c+d-3a)(c+d+a-3b)(d+a+b-3c)(a+b+c-3d)$.

18. $(b+c+d)^{-1}+(c+d+a)^{-1}+(d+a+b)^{-1}+(a+b+c)^{-1}$.

19. Σa^2bc. 20. Σa^2b^2c.

Evaluate the following symmetric functions for the equation

$$f(x) \equiv x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_n=0, a_1, a_2, a_3 \ldots a_n,$$ being its roots:

21. $\Sigma(a_1-a_2)^2$. 22. $\Sigma(a_2+a_3+a_4+\ldots+a_n)^{-1}$.

23. $\Sigma a_1/(a_2+a_3+a_4+\ldots+a_n)$. 24. $\Pi(k^2-a_1^{-2})$.

25. $\Pi(a_2a_3a_4\ldots a_n-k)$. 26. $\Pi(a_1+a_1^{-1}-2)$.

a, b are the roots of $x^2+px+q=0$ and c, d, are those of $x^2+rx+s=0$; calculate the following symmetric functions in terms of p, q, r, s:-

27. $(la+mc)(la+md)(lb+mc)(lb+md)$.

28. $(x-ac)(x-ad)(x-bc)(x-bd)$.

29. $\left\{\begin{matrix} a & b \\ c & d \end{matrix}\right\}\left\{\begin{matrix} a & b \\ d & c \end{matrix}\right\}\left\{\begin{matrix} c & d \\ a & b \end{matrix}\right\}\left\{\begin{matrix} c & d \\ b & a \end{matrix}\right\}$.

5·6. Transformation of equations.

Shew how to find the equation whose roots are $F(a_1)$, $F(a_2)$, ..., $F(a_n)$ where $F(x)$ is a rational function of x and $a_1, a_2, \ldots a_n$, are as usual, the roots of

$$f(x) \equiv p_0x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_n=0.$$

Consider the product

$$[x-F(a_1)][x-F(a_2)][x-F(a_3)]\ldots[x-F(a_n)].$$

It is easily seen that this product is a rational symmetric function of roots. Hence, by §5·51, it can be evaluated in terms of the coefficients $p_0, p_1, p_2, \ldots, p_n$. Suppose this is done and denote the resulting polynomial of the nth degree by $P(x)$; we thus have the identity

$$P(x) \equiv [x-F(a_1)][x-F(a_2)][x-F(a_3)]\ldots[x-F(a_n)].$$

The right hand product of the identity shows that $F(a_1)$, $F(a_2)$, $F(a_3), \ldots, F(a_n)$ are the roots of

$$P(x)=0.$$

2nd method: Denote $F(a_1)$, $F(a_2)$, $F(a_3), \ldots F(a_n)$ by $y_1, y_2, \ldots y_n$ respectively. Thus we have

$$y_1 = F(a_1) \quad \text{where } f(a_1) = 0. \tag{1}$$

Eliminating a_1 between equations (1), we have, suppose, $P(y_1) = 0$. This shows that y_1 or $F(a_1)$ is a root of

$$P(x) = 0 \tag{2}$$

Similarly we have

$$y_2 = F(a_2) \text{ where } f(a_2) = 0. \tag{1'}$$

Eliminating a_2 between equations (1'),—forms of functions being the same as before,—we must have $P(y_2) = 0$. This shows that y_2 or $F(a_2)$ is a root of the same equation

$$P(x) = 0. \tag{2'}$$

In this way, we can shew that $F(a_3) \ldots F(a_n)$ are the roots of

$$P(x) = 0.$$

We thus derive the general rule:

To find the equation whose roots are $F(a_1), F(a_2) \ldots F(a_n)$, *where* $a_1, a_2, \ldots, a_n$ *are the roots of* $f(x) = 0$; *eliminate* x *between* $y = F(x)$ *and* $f(x) = 0$. *The result of elimination, suppose* $P(y) = 0$, *gives the required equation.*

The method to be employed in the case where a root of the required equation is a function of two or more roots is indicated by the following particular case:

Suppose we have to form the equation whose roots are ab, ac, ad, bc, cd, db (the six products correspond to the six distinct combinations of a, b, c, d taken two at a time) where a, b, c, d are the roots of

$$f(x) \equiv x^4 + px^3 + qx^2 + rx + s = 0.$$

Consider the product

$$(x-ab)(x-cd)(x-ac)(x-bd)(x-ad)(x-bc).$$

It is easily seen that this product is a symmetric function of roots, hence, by §5·51, it can be evaluated in terms of the coefficients rationally. Denoting the resulting polynomial by $F(x)$, we have the identity

$$F(x) \equiv (x-ab)(x-cd)\ldots(x-ad)(x-bc)$$

from which we see that ab, cd,...satisfy the equation

$$F(x) = 0.$$

2nd method. Denote the six products ab, cd,...by y_1, y_2..... We have

$$y_1 = ab \text{ where } f(a) = 0 \text{ and } f(b) = 0. \tag{3}$$

Eliminating a and b between these equations we have, suppose,

$$F(y_1)=0$$

which shows that y_1 or ab is a root of

$$F(x)=0. \tag{4}$$

Similarly we have $y_2=cd$ where $f(c)=f(d)=0$. (3′)

Observing that the set of equations (3′) is identical in form, with the set of equations (3), the result of eliminating c, d, from equations (3′) must be $F(y_2)=0$.

Hence y_2 also satisfies the equations $F(x)=0$. In the same way, it can be shown that the other four products satisfy the equation (4).

We shall apply the above principles to important particular cases of transformation

5·61. *Find the equation the roots of which are* $-a_1, -a_2, \ldots -a_n$. *where* $a_1, a_2, a_3, \ldots a_n$ *are the roots of*

$$f(x)\equiv p_0x^n+p_1x^{n-1}+p_2x^{n-2}+p_3x^{n-3}+\ldots+p_n=0.$$

Applying the above rule (§ 5·6) we eliminate x between $y=-x$ and $f(x)=0$. The result is clearly $f(-y)=0$.

Writing x for y, the required equation is

$$f(-x)=0.$$

Example. If a_1, a_2, a_3 are the roots of $x^3+px^2+qx+r=0$ then $-a_1$, $-a_2$, $-a_3$ are those of $x^3-px^2+qx-r=0$.

5·62. *Find the equation whose roots are* $a_1^{-1}, a_2^{-1}, a_3^{-1}\ldots$, $a_1, a_2, a_3, \ldots$, *being the roots of the same equation as that of* § 5·61.

Applying the rule of § 5·6, we have the required equation in the form $f(x^{-1})=0$.

Multiplying by x^n, the equation can be written thus

$$p_nx^n+p_{n-1}x^{n-1}+p_{n-2}x^{n-2}+\ldots+p_0=0. \tag{1}$$

This form shows that the *required equation is derived from the given equation by writing the coefficients in the reverse order*. e. g.

Ex. The equation whose roots are the reciprocals of those of

$$x^3+bx+c=0 \text{ is } cx^3+bx^2+0x+1=0.$$

(Notice how we introduce a term with zero coefficient to account for the absent term in x^2)

5.621. Since $a_1^{-1}, a_2^{-1}, a_3^{-1}, \ldots a_n^{-1}$ are the roots of the equation (1) of § 5·62, by applying the method of §§ 5·4, 5·41, to this equation, it is possible to evaluate

$$a_1^{-m}+a_2^{-m}+a_3^{-m}+\ldots+a_n^{-m} \text{ or } S_{-m}$$

in terms of the coefficients of the equation. Thus it is easy to prove that S_{-m} *where m is a positive integer, is the coefficient of* x^n *in the quotient when* $np_n+(n-1)p_{n-1}x+\ldots+p_1x^{n-1}$ *is divided by* $p_n+p_{n-1}x+p_{n-2}x^2+\ldots+p_0x^n$ *by the process of Ascending Continued Division.*

Example 1. Prove that

(i) $p_n S_{-m}+p_{n-1}S_{-m+1}+p_{n-2}S_{-m+2}+\ldots$
$$+p_{-m+n+1}S_{-1}+mp_{-m+n}=0 \quad (m<n)$$

(ii) $p_nS_{-m}+p_{n-1}S_{-m+1}+p_{n-2}S_{-m+2}+$
$$\ldots+p_0S_{-m+n}=0 \quad (m<n)$$

Example 2. If a, b, c, d be the roots of the equation $px^4+4qx^3+6rx^2+4sx+t=0$ show that

$48(q^2-pr)=p^2\Sigma(a-b)^2$ and $48\,(s^2-rt)=p^2\Sigma[(a-b)^2c^2d^2]$

(*B. U.* 1918 *Hons.* I)

Now $\Sigma(a-b)^2=(a-b)^2+(a-c)^2+(a-d)^2+(b-c)^2+(b-d)^2+(c-d)^2$
$$=3(a^2+b^2+c^2)-2(ab+ac+ad+bc+bd+cd)$$
$$=3(a+b+c)^2-8(ab+ac+ad+bc+bd+cd)$$
$$=3(-4q/p)^2-8.6r/p=48(q^2-pr)\div p^2 \qquad (1)$$

By writing $a^{-1}, b^{-1}, c^{-1}, d^{-1}$ for a, b, c, d and therefore $t, 4s, 6r, 4q, p$ for $p, 4q, 6r, 4s, t$ (§5·62) in (1), we have

$$\Sigma(a^{-1}-b^{-1})^2=48(s^2-rt)/p^2. \qquad (2)$$

Multiplying both sides of (2) by $a^2b^2c^2d^2$ and observing that $abcd=t/p$, we have

$$\Sigma[(a-b)^2c^2d^2]=48(s^2-rt)/p^2.$$

5·622. **Reciprocal Equations.** In a *reciprocal equation*, the reciprocal of every root is also a root of the equation. We shall find some properties of such equations.

If the equation,

$$f(x)\equiv p_0x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_{n-1}x+p_n=0$$

is reciprocal, then it follows from the definition that the polynomial $f(x)$ must be a product of pairs of factors like $(x-a)\times(x-a^{-1})$

or $x^2 \mp (a+a^{-1})x+1$ and of $(x+1)^k$ and $(x-1)^m$.

The last two correspond to -1 and 1, the only two numbers which are equal to their reciprocals.

Now it is easily seen that $(x+1)^k$ may be put into the form $(x^2+2x+1)^{\frac{1}{2}k}$ or $(x+1)(x^2+2x+1)^{\frac{1}{2}(k-1)}$ according as k is even or odd.

Similarly $(x-1)^m$ may be put into the form $(x^2-2x+1)^{\frac{1}{2}m}$ or $(x-1)(x^2-2x+1)^{\frac{1}{2}(m-1)}$ according as m is even or odd.

We thus see finally that if $f(x)=0$ is a reciprocal equation, then the polynomial $f(x)$ consists of the product of factors of the form x^2+px+1, $x-1$, $x+1$, and hence, it can be exhibited in one of the following four ways:

(a) $\Pi\ (x^2+px+1)$,

(b) $(x+1)\ \Pi\ (x^2+px+1)$,

(c) $(x-1)\ \Pi\ (x^2+px+1)$,

and (d) $(x^2-1)\ \Pi\ (x^2+px+1)$.

In the case (a), the equation is of even degree and it is easy to see by arguing as in *example* 5, §1·41, that the coefficients equidistant from the two ends are identical both in magnitude and in sign.

In the same way, in the case (b), the equation is of odd degree and the coefficients counting from the beginning and the end are equal.

In the case (c), the equation is of odd degree and we may see by arguing as in *example* 5, §1·41, that the coefficients counting from the two ends, are equal in magnitude but are opposite in sign.

In the last case, the equation is of even degree and the coefficients equidistant from the two ends are equal in magnitude but opposite in sign.

Examining the four types, we find that $f(x)$ *can always be reduced to the first type by dividing it, if necessary, by* $x+1$, $x-1$, *or* x^2-1. The corresponding type is therefore regarded as the *standard type of reciprocal equations.*

It is easy to prove conversely that *if in an equation* $f(x)=0$

$$p_0=p_n,\ p_1=p_{n-1},\ p_2=p_{n-2},\ \ldots,\ p_0=p_n:$$

or $$p_0=-p_n,\ p_1=-p_{n-1},\ p_2=-p_{n-2},\cdots p_0=-p_n;$$
the equation is reciprocal.

For it follows immediately from either of the given conditions that
$$f(x)\equiv\pm x^n f(1/x),$$
and hence, by § 5·62, the reciprocal of a root of $f(x)$ is also a root of $f(x)=0$.

Lastly, *a reciprocal equation of the standard type can be transformed into one of half its degree*.

For in this case
$$f(x)\equiv\Pi(x^2+px+1)\equiv x^{\frac{1}{2}n}\Pi(x+x^{-1}+p).$$

Hence, putting y for $x+x^{-1}$, we have the transformed equation
$$\Pi(y+p)=0,$$
the degree of which is clearly $\frac{1}{2}n$.

Example 1. Solve the equation $12x^4-44x^3+63x^2-44x+12=0$.

The equation is a reciprocal equation of the standard type; hence putting $x+x^{-1}=y$, we reduce it to the form $12y^2-44y+39=0$. The roots of this quadratic are $y=3/2$ or $13/6$. The given equation is therefore equivalent to
$$x+x^{-1}=3/2 \text{ and } x+x^{-1}=13/6,$$
or
$$2x^2-3x+2=0 \text{ and } 6x^2-13x+6=0.$$
The roots of these are $(3\pm i\sqrt{7})\div 4$ and $\frac{2}{3}$, $\frac{3}{2}$.

Examples V. f.

Solve the equations:

1. $x^4-10x^3+26x^2-10x+1=0$
2. $2x^5+x^4-19x^3+19x^2-x-2=0$.
3. $x^5+x^4-18x^3-18x^2+x+1=0$
4. $x^4-3x^3-2x^2-3x+1=0$.
5. $(x+1)^4=a(x^4+1)$.
6. $(x+1)^5=a(x^5+1)$.

5·63. *Find the equation whose roots are* $ma_1, ma_2,\ldots$, *where* $a_1, a_2,\ldots$, *are the roots of* $f(x)=0$.

By the rule of §5·6, the required equation is obtained by eliminating x between $y=mx$ and $f(x)=0$. Hence the equation is
$$f(y\div m)=0.$$
Writing x for y and multiplying it by m^n, the equation reduces to
$$p_0x^n+p_1mx^{n-1}+p_2m^2x^{n-2}+p_3m^3x^{n-3}+\ldots+p_0m^n=0.$$

Hence the *required equation is derived from the given equation by multiplying the second term by m, the 3rd, by m^2 ; the 4th, by m^3; and so on.*

It is however understood that the absent terms in x are accounted for, by introducing corresponding terms with zero coefficients to make the equation complete.

e. g. To obtain the equation whose roots are ten times those of $x^4+3x^3+4x+7=0$, we write the given equation in the form, $x^4+3x^3+0x^2+4x+7=0$. Applying the rule to the latter form, we obtain the transformed equation

$$x^4+30x^3+4000x+70000=0.$$

Example 1. Transform the equation

$$x^4+\tfrac{3}{2}x^3+\tfrac{5}{4}x^2+\tfrac{1}{3}x+\tfrac{1}{9}=0$$

into one with integral coefficients and with unity as the leading coefficient.

The equation whose roots are m times those of the given equation is $x^4+\frac{3}{2}mx^3+\frac{5}{4}m^2x^2+\frac{1}{3}m^3x+\frac{1}{9}m^4=0.$

A little inspection shows that if we put $m=6$, the equation is cleared of fractions, and is reduced to

$$x^4+9x^3+45x^2+72x+144=0.$$

Examples V. g.

Transform the following equations into those with integral coefficients and with unity as the leading coefficient :

1. $8x^3+12x^2+6x+1=0.$ 2. $x^3+\frac{1}{2}x^2+\frac{1}{10}x+10=0.$
3. $9x^4+12x^3+7x^2+2x+1=0.$
4. $24x^4+20x^3+6x^2+x+\frac{1}{6}=0.$

5·64. *Find the equation whose roots are a_1-h, a_2-h, a_3-h, ... a_n-h, where a_1, a_2, ..., a_n are the roots of*

$$f(x)\equiv p_0x^n+p_1x^{n-1}+\ldots+p_n=0.$$

By the rule of § 5·6, the required equation is obtained by eliminating x between $y=x-h$ and $f(x)=0$. Hence it is given by

$$f(y+h)=0.$$

Writing x for y and applying the theorem of §5·3, the equation can be written in the form

$$f(x+h)=f(h)+\frac{1}{1!}f'(h)\,x+\frac{1}{2!}f''(h).\,x^2+\ldots+\frac{1}{n!}f^{(n)}(h).x^n=0.$$

5.641. As this transformation like the preceding one, has an important application in connexion with the Horner's method of approximation to roots, we describe below, a more practical and expeditious method of calculating the coefficients in the transformed equation.

Suppose that $b_0, b_1, b_2, \ldots b_n$ are the coefficients in the transformed equation, so that we have the identity

$$f(x+h) \equiv b_0x^n + b_1x^{n-1} + b_2x^{n-2} + b_3x^{n-3} + \ldots + b_n.$$

Writing x for $x+h$ in this, we have another identity

$$f(x) \equiv b_0(x-h)^n + b_1(x-h)^{n-1} + b_2(x-h)^{n-2} + \ldots + b_{n-1}(x-h) + b_n. \quad (1)$$

From the right hand expression, we see that when $f(x)$ is divided by $(x-h)$, the quotient is

$$b_0(x-h)^{n-1} + b_1(x-h)^{n-2} + b_2(x-h)^{n-3} + \ldots + b_{n-1}, \quad (2)$$

and the remainder, b_n

Similarly, when the expression (2) is divided, the quotient is

$$b_0(x-h)^{n-2} + b_1(x-h)^{n-3} + \ldots + b_{n-3}(x-h) + b_{n-2}, \quad (3)$$

and the remainder, b_{n-1}.

In this way, we may calculate successively all the coefficients

$$b_n, b_{n-1}, b_{n-2}, \ldots, b_0.$$

Example 1. Find the equation whose roots are those of $x^5+4x^4+3x^2+x+5=0$, diminished by 2.

Employing the Synthetic Method of Division described in §5·1, and writing only the coefficients, the following scheme shows how the transformation can be effected in practice.

1	4	0	3	1	5
0	2	12	24	54	110
1	6	12	27	55	115
0	2	16	56	166	
1	8	28	83	221	
0	2	20	96		
1	10	48	179		
0	2	24			
1	12	72			
0	2				
1	14				

Thus the required equation is

$$x^5+14x^4+72x^3+179x^2+221x+115=0.$$

Example 2. Find the equation whose roots are greater than those of $x^4-4x^3+2x^2+2x-6=0$ by 1.

Here the roots are algebraically diminished by -1; therefore the divisor is $x+1$. Writing down the coefficients and applying the method of Synthetic Division, we have the following scheme:

	1	−4	2	2	−6
0		−1	5	−7	5
1		−5	7	−5	−1
0		−1	6	−13	
1		−6	13	−18	
0		−1	7		
1		−7	20		
0		−1			
1		−8			

Thus the required equation is

$$x^4-8x^3+20x^2-18x-1=0.$$

Example 3. Transform the equation

$$F(x)\equiv a_0x^3+3a_1x^2+3a_2x+a_3=0$$

so that the resulting equation wants the second term.

Diminish the roots of the equation by h; so that the transformed equation is

$$F(x+h)\equiv F(h)+\frac{x}{1!}F'(h)+\frac{x^2}{2!}F''(h)+\frac{x^3}{3!}F'''(h)=0. \quad (1)$$

Now choose h so that $\frac{1}{2!}F''(h)\equiv 3(a_0h+a_1)=0$ *i. e.* $h=-a_1/a_0$.

Substituting this value of h in (1), it reduces to the desired form.

It is possible, in this way, to transform an equation so that the resulting equation is wanting in the second term.

Examples V. h.

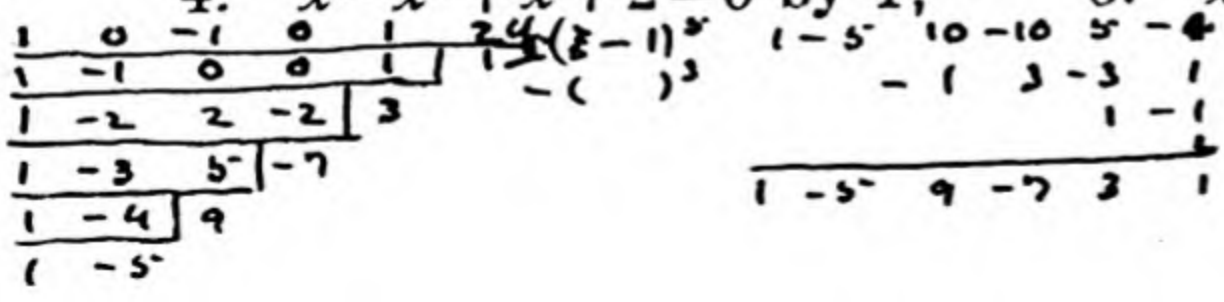
Obtain the equations whose roots are less than those of

1. $4x^4-3x^2+2x-1=0$ by 1; 2. $x^5-4x^4+3x-2=0$ by 2;

and 3. $x^6-x^2+2=0$ by 3.

Obtain the equations whose roots are greater than those of

4. $x^5-x^3+x+2=0$ by 1; 5. $x^4+3x^2-2x+7=0$ by 1.



and 6. $x^4-4x^3+2x^2-7x+5=0$ by 2

Transform the following equations so that the resulting equations do not posses the second term:

7. $x^3-3x^2+1=0$. 8. $4x^4-12x^3+7x^2-1=0$.

9. The roots of the equation, $x^3+3x^2-9x+1=0$ are increased by h. If the resulting equation has no term in x, prove that h is either 3 or -1.

10. If a and b be two quantities such that if the roots of the equation $x^4+4px^3+6qx^2+4rx+s=0$ be increased by either, the transformed equation does not involve x^2, prove that

$$a+b=2p,\ ab=q. \qquad [B.\ U.\ 1920\ Hons.\ \text{I.}]$$

Example 1. If the roots of the equation $x^3+px^2+qx+r=0$ be a, b, c; form the equation of which the roots are $a(b+c)^{-1}$, $b(c+a)^{-1}$, $c(a+b)^{-1}$. [*B. U. 1925 Pass.* I.]

Since $a+b+c=-p$, $a(b+c)^{-1}=-a(a+p)^{-1}$.

Similarly $b(c+a)^{-1}=-b(b+p)^{-1}$, and $c(a+b)^{-1}=-c(c+p)^{-1}$.

According to the rule of § 5·6, we eliminate x between $y=-x(p+x)^{-1}$ and the given equation. Hence the required equation is

$$p^3y^3-p^3y^2(1+y)+pqy(1+y)^2-r(1+y)^3=0.$$

Example 2. If a, b, c, are the roots of $x^3+qx+r=0$; form the equation whose roots are

$$\frac{b}{c}+\frac{c}{b},\ \frac{c}{a}+\frac{a}{c},\ \frac{a}{b}+\frac{b}{a}. \qquad [B.\ U.\ 1924\ Pass.\ \text{I.}]$$

Now $a+b+c=0$ and $abc=-r$.

$\therefore\ bc^{-1}+cb^{-1}=(b^2+c^2)/bc=[(b+c)^2-2bc]/bc=(a^2-2bc)/bc$

$=(a^3+2r)/(-r)=-(r-qa)/r.\quad (\because\ a^3+qa+r=0)$

Similarly $ca^{-1}+ac^{-1}=-(r-qb)/r$, $ab^{-1}+ba^{-1}=-(r-qc)/r$.

Therefore by the rule of § 5·6, we eliminate x between $y=-(r-qx)/r$ and the given equation; *i. e.* between $x=r(1+y)/q$ and the equation.

Hance the required equation is

$$r^2(1+y)^3+q^3(1+y)+q^3=0.$$

Example 3. If a, b, c are the roots of $px^3+3qx^2+3rx+s=0$, prove that the equation whose roots are

$$\frac{bc-a^2}{b+c-2a},\ \frac{ca-b^2}{c+a-2b},\ \frac{ab-c^2}{a+b-2c},$$

is $\dfrac{\text{a homographic transform}}{fxy+g(x+y)+r=0}$,

(Harley 95(6))

$$p(qx+r)^3-3q(qx+r)^2(px+q)+3r(qx+r)(px+q)^2-s(px+q)^3=0$$

[*B. U.* 1917 *Hons.* I]

Since $a+b+c=-3q/p$, $bc+ca+ab=3r/p$, $abc=-s/p$,

$$\frac{bc-a^2}{b+c-2a}=\frac{abc-a^3}{a(a+b+c-3a)}=\frac{-s-pa^3}{a(-3q-3pa)}=\frac{pa^3+s}{3pa^2+3qa}.$$

Now $pa^3+3qa^2+3ra+s=0$; $\therefore pa^3+s=-3qa^2-3ra$.

$$\therefore \frac{bc-a^2}{b+c-2a}=\frac{-3qa^2-3ra}{3pa^2+3qa}=-\frac{qa+r}{pa+q}.$$

Similarly $\dfrac{ca-b^2}{c+a-2b}=-\dfrac{qb+r}{pb+q}$, $\dfrac{ab-c^2}{a+b-2c}=-\dfrac{qc+r}{pc+q}$.

Hence by the rule of § 5·6, the required equation is obtained by eliminating x between $y=-(qx+r)\div(px+q)$ and the given equation.

Now if $y=-(qx+r)\div(px+q)$, then $x=-(qy+r)\div(py+q)$.

The required equation is therefore derived from the given equation by putting $-(qy+r)\div(py+q)$ for x and then writing x for y.

N. B. Mark how each of the roots of the required equation is transformed and expressed as a function of a single root of the given equation.

Example 4. If a, b, c are the roots of the equation $x^3+x^2+x-k=0$, prove that $a^3b+b^3c+c^3a$ and $ab^3+bc^3+ca^3$ are the roots of $x^2+(1-k)x+7k^2+3k+1=0$. [*B. U. 1916 Hons.* I.]

Denote $a^3b+b^3c+c^3a$, $ab^3+bc^3+ca^3$ by l and m so that the required equation is $(x-l)(x-m)\equiv x^2-(l+m)x+lm=0$.

Now $l+m=a^3b+b^3c+c^3a+ab^3+bc^3+ca^3=\Sigma ab\,\Sigma a^2-abc\,\Sigma a$.

$$\therefore\ l+m=1.(1-2)-(+k)(-1)=k-1;$$

and
$$lm=(a^3b+b^3c+c^3a)(ab^3+bc^3+ca^3)$$
$$=abc(a^5+b^5+c^5)+a^2b^2c^2(bc+ca+ab)+b^4c^4+c^4a^4+a^4b^4$$
$$=7k^2+3k+1.$$

Hence the required equation is $x^2+(1-k)x+7k^2+3k+1=0$.

N. B. In this case the "symmetric-function-method" seems to to be the only method applicable.

Example 5. Find the condition that the difference between two of the roots of the equation $x^3+3px+q=0$ may be h.

If a, b, c are the roots of the equation, then the condition requires that one of the six quantities $b-c\pm h$, $c-a\pm h$, $a-b\pm h$ must vanish

$$i.\,e.\ F(h)\equiv[h^2-(b-c)^2][h^2-(c-a)^2][h^2-(a-b)^2]=0.$$

Now, from the equation, $a+b+c=0$ and $abc=-q$.

Hence, $(b-c)^2=(b+c)^2-4bc=a^2-4bca.a^{-1}=a^2+4qa^{-1}$; but $a^3+3pa+q=0$, $\therefore \underline{(b-c)^2}=a^2+4qa^{-1}=(a^3+4q)a^{-1}=\underline{3(q-pa)a^{-1}}$

Similarly $(c-a)^2=3(q-pb)b^{-1}$, and $(a-b)^2=3(q-pc)c^{-1}$.

$\therefore F(h)=(h^2+3p-3qa^{-1})(h^2+3p-3qb^{-1})(h^2+3p-3qc^{-1})$

$=(h^2+3p)^3-3q(h^2+3p)^2\Sigma a^{-1}+9q^2(h^2+3p)\Sigma a^{-1}b^{-1}-27q^3a^{-1}b^{-1}c^{-1}.$

$=(h^2+3p)^3+9p(h^2+3p)^2+27q^2.$

Hence the necessary and sufficient condition that the difference between two roots may be h is $(h^2+3p)^3+9p(h^2+3p)^2+27q^2=0$.

N. B. Note how the condition is expressed as a symmetric function of the roots.

Example 6. Find the condition that the sum of two of the roots of $x^5+px^4+qx^3+rx+s=0$ may be zero. Prove further, that the symmetric function $\Pi(a+b)$ is a multiple of $s^2-pqs+q^2r$, a, b being two of the roots of the given equation.

Since the sum of two of the roots is zero, they must be equal in magnitude but opposite in sign.

Hence $x^5+px^3+qx^2+rx+s=0$ (1)

and $x^5+px^3-qx^2+rx-s=0$ (2)

must have a common root.

This root must be common to

$(1)+(2)\equiv 2x(x^4+px^2+r)=0$ (3)

and $(1)-(2)\equiv 2(qx^2+s)=0.$ (4)

Since zero is not a common root of (1) and (2), the common root of (1) and (2) must be common to

$x^4+px^2+r=0$ and $qx^2+s=0.$ (5)

Hence eliminating x, between equations (5), we have the necessary condition

$s^2/q^2-ps/q+r=0$ or $s^2-pqs+q^2r=0.$

Now the condition expressed in terms of the roots is

$$\Pi(a+b)=0,$$

where $\Pi(a+b)$ is a symmetric homogeneous function of ten dimensions. Hence $\Pi(a+b)$ must be a multiple of $s^2-pqs+q^2r$.

Otherwise:-Assume that

$(x^2+k)(x^3+lx^2+mx+n)\equiv x^5+px^3+qx^2+rx+s;$

then $l=0$, $m+k=p$, $n+kl=q$, $mk=r$, $nk=s$; eliminating k, l, m, n etc.

Examples. V. i.

It a, b, c are the roots of $f(x) \equiv x^3+3px+q=0$; find the equation whose roots are the different functions of the roots of the form,

1. $(b^2+c^2)a^{-1}$: 2. bc; 3. $a^2(b+c)^{-1}$; 4. $bc(b+c)^{-1}$;
5. $bc-a^2$; 6. bc^{-1}; 7. $bc^{-1}+b^{-1}c$; 8. $(b-c)^2$;
9. $(b+c-bc)(b+c+bc)^{-1}$.

Ii a, b, c, d are the roots of $f(x) \equiv x^4+px^3+qx^2+rx+s=0$, find the equation whose roots are all the values of

10. $b+c+d$; 11. $(1+a)(1-a)^{-1}$; 12. $bcd(b^{-1}+c^{-1}+d^{-1})$.

If $a_1, a_2, \ldots a_n$ are the roots of $f(x) \equiv x^n+p_1x^{n-1}+\ldots+p_n=0$, find the equation whose roots are all the values of,

13. $(a_2+a_3+\ldots+a_n)a_1^{-1}+a_2a_3\ldots a_n$;
14. $(a_2^{-1}+a_3^{-1}+\ldots+a_n^{-1})$;
15. $a_1^{-1}(a_2+a_3+\ldots+a_n)(a_2^{-1}+a_3^{-1}+\ldots+a_n^{-1})$.

16. Prove that the equation whose roots are $a_1^2, a_2^2, a_3^2, \ldots$, where $a_1, a_2, \ldots, a_n$, are the roots of $f(x) \equiv x^n+p_1x^{n-1}+\ldots+p_n=0$, is

$$(p_n+p_{n-2}x+p_{n-4}x^2+\ldots)^2-x(p_{n-1}+p_{n-3}x+p_{n-5}x^2+\ldots)^2=0,$$

or

$$f(x^{\frac{1}{2}})f(-x^{\frac{1}{2}})=0.$$

17. If $\sqrt{a_1}+\sqrt{a_2}+\ldots+\sqrt{a_5}=0$

and $$1/(\sqrt{a_1})+1/(\sqrt{a_2})+\ldots+1/(\sqrt{a_5})=0,$$

prove that $(\Sigma a_r)^2=4\Sigma a_ra_s$ and $(\Sigma a_r^{-1})^2=4\Sigma a_r^{-1}a_s^{-1}$.

[*B. U. 1924 Problems.*]

[The equation whose roots are $\sqrt{a_1}, \sqrt{a_2}, \ldots$, must be of the form $x^5+bx^3+cx^2+e=0$. The equation whose roots are the squares of the roots of this equation must be $(e+cx)^2-x^3(b+x)^2=0$; Hence

$$(\Sigma a_r)^2=4\Sigma a_ra_s=4b^2, etc.\ ]$$

5·7. Depression of the degree of an equation when a relation between two of its roots is known. *When two of the roots of an equation are connected by a relation, they can be found and the degree of the equation, be reduced by two.*

Let $f(x)=0$ be the equation and a, b two of its roots connected by a relation of form $a=F(b)$. Since a, b are roots of $f(x)=0$,

$$\therefore\ f(a)=0, \text{ and } f(b)=0. \tag{1}$$

Now $a=F(b)$, $\therefore$ from (1), $f[F(b)]=0$. (2)

From (1) and (2), we find that b is a common root of

$$f(x)=0, \text{ and } f[F(x)]=0. \tag{3}$$

$\therefore$ $x-b$ must be a common factor of $f(x)$ and $f[F(x)]$. This common factor can be found by applying the familiar process of finding the *H. C. F.* to $f(x)$ and $f[F(x)]$. Substituting the value of b so determined in $a=F(b)$, we get the value of a Now divide $f(x)$ by the product $(x-a)(x-b)$ and equate the quotient to zero. The resulting equation is of a lower degree than the given equation by two.

In the above argument, we have implicitly supposed $f(x)$ to be a polynomial. With some modification the argument may be made more general.

Example 1. If the product of two of the roots of the equation $4x^4-16x^3+31x^2-30x+14=0$ is 2, solve it.

[*B. U. 1922 Hons I.*]

Let a, b be two of the roots of the given equation connected by the relation $ab=2$ or by $a=2b^{-1}$. Now we have

$4a^4-16a^3+31a^2-30a+14=0$ and $4b^4-16b^3+31b^2-30b+14=0$;

$\therefore$ $64b^{-4}-128b^{-3}+124b^{-2}-60b^{-1}+14=0$

or $14b^4-60b^3+124b^2-128b+64=0$.

Hence the factor $x-b$ is common to $4x^4-16x^3+31x^2-30x+14$ and $14x^4-60x^3+124x^2-128x+64$.

The *H. C. F.* of these is x^2-2x+2.

Dividing $4x^4-16x^3+31x^2-30x+14$ by x^2-2x+2, the other factor is found to be $4x^2-8x+7$. The given equation is therefore equivalent to the equations $x^2-2x+2=0$, and $4x^2-8x+7=0$.

$\therefore$ the roots of the given equation are $1\pm i$, $1\pm i\frac{1}{2}\sqrt{3}$.

Otherwise:–The quadratic equation whose roots are a, b must be of the form $x^2+px+2=0$.

Hence the other factor of $4x^4-16x^3+31x^2-30x+14=0$ must be of the form $4x^2+qx+7$. Thus we have an identity of the form

$$4x^4-16x^3+31x^2-30x+14\equiv(x^2+px+2)(4x^2+qx+7).$$

Equating coefficients of like powers on both sides, we get

$$4p+q=-16,\ 7+pq+8=31,\ 7p+2q=-30.$$

$\therefore$ $p=-2$ and $q=-8$. *etc.*

Example 2. Two roots of the equation

$$x^5+2x^4+5x^3+3x^2+6x+4=0$$

are connected by the relation $a^2+ab+b^2=0$; find them.

[*B. U. 1920 Problems.*]

Since two of the roots a, b are connected by the relation $a^2+ab+b^2=0$, the corresponding quadratic factor must be of the form x^2+px+p^2. Therefore the other factor of

$$x^5+2x^4+5x^3+3x^2+6x+4$$

must be of the from x^3+qx^2+rx+4/p^2.

Hence we have an identity of the form

$$x^5+2x^4+5x^3+3x^2+6x+4\equiv(x^2+px+p^2)(x^3+qx^2+rx+4/p^2).$$

Equating the coefficients of like powers of x, we have

$$p+q=2 \quad (1); \qquad r+pq+p^2=5 \quad (2);$$

$$4/p^2+pr+p^2q=3 \quad (3); \qquad 4/p+p^2r=6 \quad (4).$$

From these, we have

$$2p^4-5p^3+6p-4=0 \text{ and } p^4-2p^3+3p-6=0.$$

The common root is easily found to be 2; the corresponding values of q, r are 0, 1. $\therefore$ the given equation is equivalent to

$$x^2+2x+4=0 \text{ and } x^3+x+1=0.$$

The roots of the first equation are $2w, 2w^2$ where w is a complex cube root of unity. These are the values of a and b.

Examples. V. j.

1. Solve the equation $x^4-6x^3+3x^2+12x+4=0$, two of its roots a, b being connected by $a=2b$.

2. Solve the equation $x^4-2x^3-7x^2+8x-2=0$, two of the roots a, b being connected by $b=3+a$.

3. Solve the equation $x^5-28x^3-40x^2+31x-4=0$, two of the roots a, b being connected by the relation $ab=a+2$.

4. Prove that the equation $x^5+ax^4+bx+c=0$ will have the sum of three of its roots equal to zero, provided $(ab+c)^2=ca^5$, and can then be reduced to a quadratic and a cubic.

Apply this method to solve the equation:

$$x^5+2x^4+3x+18=0. \qquad [B.\ U.\ M.\ A.\ 1919.]$$

[Assume that $x^5+ax^4+bx+c\equiv(x^3+lx+m)(x^2+px+c/m)$ so that $p=a$, $c/m+l=0$, $lp+m=0$, $lc/m+mp=b$...]

5. The equation $x^4-2x^3+4x^2+6x-21=0$ has two roots equal in magnitude and opposite in sign; find all the roots.

[*B. U. 1925 Hons I*]

5·8. Cubic and Quartic Equations.

We know that the roots of the general equation of the 2nd degree $ax^2+bx+c=0$, are given by the formulae

$$[-b \pm \sqrt{(b^2-4ac)}]/2a.$$

By substituting for a, b, c, we may obtain the roots of any quadratic. Similar formulae are obtained for the roots of equations of the 3rd and 4th degrees below.

5·81. We have seen (§ 5·641, *Example* 3) that by increasing the roots of any cubic by a suitable quantity, we can reduce it to a form, not containing the term in x^2. Hence we may attempt, without loss of generality, the solution of the equation given in the form

$$x^3+3qx+r=0.$$

We have the following elementary identity

$$x^3-3abx+a^3+b^3 \equiv (x+a+b)(x+a\omega+b\omega^2)(x+a\omega^2+b\omega), \quad \ldots\ldots(1)$$

where ω is a complex cube root of unity, (§ 5·411)

Now, assuming the identity, (§ 5·411)

$$x^3+3qx+r \equiv x^3-3abx+a^3+b^3$$
$$\equiv (x+a+b)(x+a\omega+b\omega^2)(x+a\omega^2+b\omega) \quad \ldots\ldots(4)$$

and equating the coefficients of like powers, we have

$$3q=-3ab, \text{ and } r=a^3+b^3. \quad \ldots\ldots(3)$$

$\therefore a^3$ and b^3 are the roots of the quadratic $t^2-rt-q^3=0$ the roots of which are

$$[r \mp \sqrt{(r^2+4q^3)}]/2.$$

These are the values of a^3, b^3. Denoting the cube roots of these—whose product is $-q$,—by l and m, we see from (4), that the roots of the given equation are

$$-l-m, \quad -l\omega-m\omega^2, \quad -l\omega^2-m\omega. \quad \ldots\ldots(5)$$

These formulae for the roots of the cubic are not very handy and we rarely use them in practice.

5·811. So far we have imposed no restrictions upon q and r. They could be real or complex. In the sequel, we assume them to be real numbers. We then easily verify the following statements:

I. If r^2+4q^3 is positive, one of the roots is real and the remaining two are conjugate complex numbers.

II. If r^2+4q^3 is zero, all the roots are real, two of them being equal.

III. If r^2+4q^3 is negative, all the roots are real although

they appear to be complex. This case is not very obvious; we therefore prove it formally.

Since $r^2+4q^3<0$, q must be negative; $\therefore\ -r^2/4q^3<1$.

Hence, it is possible to find an acute angle, θ such that

$$\cos\theta = r \div 2(-q)^{\frac{3}{2}}.$$

Making this substitution for r in the roots (5), we have

$$-l-m = -[\tfrac{1}{2}r+\tfrac{1}{2}\sqrt{(r^2+4q^3)}]^{\frac{1}{3}}-[\tfrac{1}{2}r-\tfrac{1}{2}\sqrt{(r^2+4q^3)}]^{\frac{1}{3}}$$

$$=-(-q)^{\frac{1}{2}}[\cos\tfrac{1}{3}\theta+i\sin\tfrac{1}{3}\theta+\cos\tfrac{1}{3}\theta-i\sin\tfrac{1}{3}\theta]=-2(-q)^{\frac{1}{2}}\cos\tfrac{1}{3}\theta.$$

Remembering that $w=\cos\frac{2}{3}\pi+i\sin\frac{2}{3}\pi$ (see § 5·411), we get similarly

$$-lw-mw^2=-2(-q)^{\frac{1}{2}}\cos\tfrac{1}{3}(\theta+2\pi) \text{ and } -lw^2-mw=-2(-q)^{\frac{1}{2}}\cos\tfrac{1}{3}(\theta+4\pi).$$

The expression r^2+4q^3 is called the *discriminant of the cubic*

$$x^3+3qx+r=0.$$

Example 1. Solve the equation $x^3-6x+9=0$.

Assuming the identity $x^3-6x+9\equiv x^3-3abx+a^3+b^3$,

we have $ab=2$, and $a^3+b^3=9$.

$\therefore\ a^3, b^3$ are the roots of $t^2-9t+8\equiv(t-1)(t-8)=0$. Hence a^3, b^3 are 1 and 8; $\therefore\ a, b$ are equal to 1, 2; and since the product of these cube roots is 2, the roots of the given equation are

$$-3,\ -2w-w^2,\ -2w^2-w.$$

Example 2. Solve the equation $x^3-3x^2+9x-14=0$.

Diminishing the roots of the equation by 1, (§ 5·641 *Example* 3)? it reduces to the required form $x^3+6x-7=0$. Assuming the identity

$$x^3+6x-7\equiv x^3-3abx+a^3+b^3,$$

we have $ab=-2$ and $a^3+b^3=-7$.

$\therefore\ a^3, b^3$ are the roots of $t^2+7t-8\equiv(t+8)(t-1)=0$;

$\therefore\ a^3, b^3$ are equal to -8, 1; $\therefore\ a, b$ are equal to -2, 1. Since the product of these cube roots is -2, the roots of the transformed cubic are

$$1,\ 2w^2-w,\ 2w-w^2.$$

Hence the roots of the original cubic are

$$2,\ \ 1-w+2w^2,\ \ 1+2w-w^2.$$

Example. 3. Solve the equation $x^3-3x+1=0$.

Here $q=-1$ and $r=1$. Hence the discriminant $=1-4=-3$ which is negative. Applying therfore (III) of §5·811, we get the roots

$$-2\cos\tfrac{1}{3}\theta, -2\cos\tfrac{1}{3}(\theta+2\pi), -2\cos\tfrac{1}{3}(\theta+4\pi)$$

where θ is given by $\cos\theta=1/2$ *i. e.* $\theta=60^0$.

Example. 4. If $x^3+3px+q=0$ can be put into the form $a(x-b)^3=b(x-a)^3$, prove that a, b satisfy $pt^2+qt-p^2=0$.

Solve the equation $x^3-24x+72=0$.

By hypothesis, we have the identity,

$$x^3+3px+q\equiv(a-b)^{-1}[a(x-b)^3-b(x-a)^3]^*.$$
$$=x^3-3abx+ab(a+b).$$

$\therefore$ $ab=-p$ and $ab(a+b)=q$; so that $a+b=-q/p$.

Hence a,b are the roots of $pt^2+qt-p^2=0$.

In the case of the equation $x^3-24x+72=0$ a,b are determined by, $t^2-9t+8=0$. Hence $a=8$ and $b=1$.

$\therefore$ the given equation can be thrown into the from

$$8(x-1)^3-(x-8)^3=0.$$

Hence the required roots are given by

$$2(x-1)=(x-8),\ 2(x-1)=(x-8)w,\ 2(x-1)=(x-8)w^2,$$

or by

$$-6,\ 2-2w,\ 2-2w^2.$$

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* We have (implicitly) assumed that a, b are unequal and so $q^2+4p^3\neq0$. When $q^2+4p^3=0$ the given equation has two equal roots by (II) of §5·811.

Examples. V. k.

Solve the cubic equations:

1. $x^3-6x-9=0$. 2. $x^3-6wx-9=0$. 3. $x^3-6w^2x-9=0$.

(In 2. and 3. w denotes a complex cube root of unity.)

4. $x^3+6x+7=0$, 5. $x^3+2x^2+2x+1=0$. 6. $x^3-2x^2-9=0$,

7. $7x^3+6x^2+1=0$. 8. $x^3+6x^2+11x+6=0$.

9. $x^3+6x-9=0$. 10. $x^3-12x+64=0$.

11. $x^3-6x+4=0$. 12. $x^3-3b^2x+2b^3\cos\theta=0$.

Prove the identities:

13. $x^3-9x+9=(x+2.3^{\frac{1}{2}}\cos\tfrac{1}{18}\pi)$

$(x+2.3^{\frac{1}{2}}\cos\tfrac{1}{1}\tfrac{1}{8}\pi)(x+2.3^{\frac{1}{2}}\cos\tfrac{1}{1}\tfrac{3}{8}\pi)$.

14. $x^3+6x^2+9x+4\cos^2 p=[x+4\cos^2\frac{1}{3}p][x+4\cos^2\frac{1}{3}(p+\pi)]$
$\times[x+4\cos^2\frac{1}{3}(p-\pi)]$.

Express the following polynomials as the sum or difference of two cubes:

15. x^3-6x-6. 16. $2x^3+9x^2+15x+9$.

17. $9x^3+18x^2y+18xy^2+9y^3$.

5·82. We shall now attempt the solution of the general **equation of the fourth degree,**

$$x^4+ax^3+bx^2+cx+d=0. \quad (1)$$

It is easily seen that the equation (1) is equivalent to

$$(x^2+\tfrac{1}{2}ax+y)^2=x^2(2y+\tfrac{1}{4}a^2-b)+x(ay-c)+y^2-d. \quad (2)$$

for all values of y. Now choose y so that the right hand quadratic expression of (2) becomes a perfect square. y must therefore satisfy the cubic

$$(ay-c)^2=4(2y+\tfrac{1}{4}a^2-b)(y^2-d). \quad (3)$$

This cubic can be solved always, by the method of § 5·81, or occasionally, by inspection.

Substituting one of the roots, say, l of equation (3) in equation (2), the latter reduces to the form

$$(x^2+\tfrac{1}{2}ax+l)^2=(mx+n)^2. \quad (4)$$

From (4), we see that (1) is equivalent to the two quadratic equations

$$x^2+\tfrac{1}{2}ax+l=mx+n, \quad \text{and} \quad x^2+\tfrac{1}{2}ax+l=-mx-n. \quad (5)$$

Solving these quadratics, we obtain the solution of the given quartic equation.

Example. 1. Solve $x^4+2x^3+9x^2+8x+7=0$.

The given equation is equivalent to

$$(x^2+x+y)^2=x^2(2y-8)+x(2y-8)+y^2-7. \quad (1)$$

The right hand quadratic expression of (1), reduces to a square if

$$(2y-8)^2=4(2y-8)(y^2-7). \quad (2)$$

Now one of the roots of (2) is clearly 4. Substituting this value of y in (1), we reduce it to

$$(x^2+x+4)^2=9. \quad (3)$$

The equation (3) is equivalent to

$$x^2+x+4=3 \quad \text{and} \quad x^2+x+4=-3;$$

or to $x^2+x+1=0$ and $x^2+x+7=0$. (4)

Now the roots of these quadratic equations are

$$(-1\pm i\sqrt{3})/2 \quad \text{and} \quad (-1\mp 3i\sqrt{3})/2.$$

The student should satisfy himself that he gets the same set of roots, by using the remaining two roots of (2).

Example 2. Exhibit the quartic $x^4-8x^3+15x^2+4x-2$ as a difference of squares in three different ways. [*B. U. 1914 Problems*].

We have the identity

$$x^4-8x^3+15x^2+4x-2\equiv(x^2-4x+y)^2 - (x^2+2yx^2-8yx+y^2-4x+2) \quad (1)$$

Now choose y so that $x^2(1+2y)-x(8y+4)+y^2+2$ is a perfect square. y must therefore satisfy the equation

$$(2+4y)^2=(1+2y)(y^2+2). \quad (2)$$

One of the roots of (2) is obviously $-\frac{1}{2}$. The other two roots satisfy the quadratic $4(1+2y)=y^2+2$ or $y^2-8y-2=0$.
Hence they are $4+3\sqrt{2}$ and $4-3\sqrt{2}$.

Substituting these values of y in (1) we have

$$x^4-8x^3+15x^2+4x-2=(x^2-4x-\tfrac{1}{2})^2-(3/2)^2$$
$$=(x^2-4x+4\mp 3\sqrt{2})^2-[(\sqrt{6}\mp\sqrt{3})(x-2)]^2.$$

Example 3. If the equation $x^4+6ax^2+4bx+c=0$ can be written in the form $m(x-n)^4-n(x-m)^4=0$, show that $a^3+b^2=ac$ and that m and n are roots of the equation $az^2+bz-a^2=0$.

Hence or otherwise solve the equation $x^4+12x^2+8x+6=0$.

[*B. U. 1925 Problems*]

By hypothesis, we have the identity

$$x^4+6ax^2+4bx+c=[m(x-n)^4-n(x-m)^4]/(m-n)$$
$$=x^4-6mn\,x^2+4\,m\,n\,(m+n)x-m\,n(m^2+m\,n+n^2).$$

Hence, equating the coefficients of like powers, we have

$$a=-m\,n;\ b=m\,n(m+n);\ \text{and}\ c=-m\,n(m^2+m\,n+n^2).$$

From these three relations, we have $a^3+b^2=ac$.

Further, from the first two of them, we have $a=-m\,n$ and $-b/a=m+n$; hence, m, n satisfy $az^2+bz-a^2=0$.

In the case of $x^4+12x^2+8x+6=0$,

$$a=2,\ b=2,\ \text{and}\ c=6;\ \text{hence}\ a^3+b^2=a\,c.$$

So the above method is applicable. The quadratic determining m, n is $z^2+z-2=0$. The roots of this are -2, 1. Therefore

the given equation can be put into the form

$$(x+2)^4=-2(x-1)^4=2(x-1)^4[\cos(2r+1)\pi+i\sin(2r+1)\pi].$$

The roots of the given equation are therefore given by

$$x+2=2^{\frac{1}{4}}(x-1)\,[\cos(2r+1)\tfrac{1}{4}\pi+i\,\sin(2r+1)\tfrac{1}{4}\pi],\ r=0,\ 1,\ 2,\ 3.$$

Example 4. If an equation $x^4+4px^3+6qx^2+4rx+s=0$, whose roots are a, b, c, d, can be put into the form

$$(x^2+2px+l)^2=(mx+n)^2;$$

prove that

I. the possible values of $2l$ are $ab+cd;\ ac+bd,\ ad+bc$;

II. the possible values of $4n^2$ are

$$(ab-cd)^2,\ (ac-bd)^2,\ (ad-bc)^2;$$

and that III. the possible values of $4m^2$ are

$$(a+b-c-d)^2,\ (a+c-b-d)^2,\ (a+d-b-c)^2.$$

The given equation is equivalent to the two quadratic equations

$$x^2+2px+l=+(mx+n), \tag{1}$$

and

$$x^2+2px+l=-(mx+n). \tag{2}$$

If we suppose that a, b are the roots of (1), then c, d must be the roots of (2).

$\therefore\ l-n=ab$ and $l+n=cd$; $\therefore\ 2l=ab+cd$, and $-2n=ab-cd$

$$\therefore\quad 4n^2=(ab-cd)^2.$$

Again from (1), $-2p+m=a+b$ and from (2), $-2p-m=c+d$

$$\therefore\quad 2m=(a+b-c-d);\quad \therefore\quad 4m^2=(a+b-c-d)^2.$$

There are two other ways of distributing the four roots between the equations (1) and (2). Corresponding to these ways, we get two more possible values for each of $2l$, $4n^2$, and $4m^2$.

5·83. The student must have noticed that the solution of a quadratic depends upon that of a simple equation, that of a cubic, on the solution of a quadratic and the solution of a biquadratic, upon that of a cubic. Mathematicians were thus led to hope that the solution of the 5th degree could be effected by solving an auxiliary biquadratic equation. They were however surprised to find that the solution depended upon that of an equation of the sixth degree. The problem of obtaining an algebraic solution involving a finite number of algebraic operations, of equations of the fifth and

higher digrees, was finally disposed of, by Abel (1802–1829), a brilliant Norwegian Mathematician. He proves that no such solution exists.

Examples. V. l

Solve the equations:

1. $x^4+x^3-2x^2-x+1=0$, 2. $x^4-x^3-x-1=0$.

3. $x^4-5x^3+5x^2+5x-6=0$, 4. $x^4-3i\,x^3-x^2-3i\,x-2=0$.

Exhibit each of the following polynomials in the form of the sum or difference of two squares in three ways:

5. x^4+x^2-2x. 6. $x^4+2x^3+2x^2+4x-1$.

7. $x^4+4x^3+2x^2+4x+1$. 8. $x^4-3x^3+7x^2+6x-8$.

9. $(x+1)(x+2)(x+3)(x+4)$.

Solve the simultaneous equations:

10. $x^2-2xy-3y^2+x+3y+1=0=x^2-y^2+2x-3y+1$.

11. $x^3+y^3-3xy+1=0=x^2-y^2+3x-2y+1$.

Examples. V. m.

1. Prove that $(2x+1)^{2m}-4x^2-4x-1$ is divisible by $x(x+1)(2x+1)^2$.

Find the quotient.

3. Prove that $x^{3n}+x^2+x$, $x^{3n+1}+x^2+1$, $x^{3n+2}+x+1$ are divisible by x^2+x+1. Find the quotient in each case. (m, n are any integers.)

4. Prove that the remainder when

$$(x^2+x)^k+(1+x)^k+(1+x^2)^k$$

is divided by $1+x+x^2$, is 0, or $3(-1)^k$, according as k is not, or is a multiple of 3.

5. Establish the identity

$$f(a)\,\frac{(x-b)(x-c)^2}{(a-b)(a-c)^2}+f(b)\,\frac{(x-c)(x-a)^2}{(b-c)(b-a)^2}+f(c)\,\frac{(x-a)(x-b)^2}{(c-a)(c-b)^2}$$

$$\equiv f(x)+(x-a)(x-b)(x-c)\left\{\frac{f(a)}{(a-b)(a-c)^2}+\frac{f(b)}{(b-c)(b-a)^2}+\frac{f(c)}{(c-a)(c-b)^2}\right\},$$

where $f(x)$ is a polynomial of the 2nd degree.

6. Establish the identity

$$\begin{vmatrix} x & a & a & a \\ a & x & b & b \\ b & b & x & c \\ c & c & c & x \end{vmatrix} \equiv (x-a)(x-b)(x-c)(x+a+b+c).$$

7. If $a+b+c=0$, then $4(a^7+b^7+c^7)=7abc(a^2+b^2+c^2)^2$.

Deduce the identities

$$4[(y-z)^7+(z-x)^7+(x-y)^7] = 7(y-z)(z-x)(x-y)[(y-z)^2+(z-x)^2+(x-y)^2]^2.$$

$$4[x^7(y-z)^7+y^7(z-x)^7+z^7(x-y)^7] = 7xyz(y-z)(z-x)(x-y)[x^2(y-z)^2+y^2(z-x)^2+z^2(x-y)^2]^2.$$

8. If $a+b+c+d+e=a^{-1}+b^{-1}+c^{-1}+d^{-1}+e^{-1}=0$, prove that

$$2(a^4+b^4+c^4+d^4+e^4)=(a^2+b^2+c^2+d^2+e^2)^2$$

and $$2(a^{-4}+b^{-4}+c^{-4}+d^{-4}+e^{-4})=(a^{-2}+b^{-2}+c^{-2}+d^{-2}+e^{-2})^2.$$

9. In a reciprocal equation

$$S_m = S_{-m}$$

where S_r denotes the sum of the rth powers of roots; prove the converse also.

10. Show that the sum of the mth powers of the roots of the equation $x^n-x^{n-1}-x^{n-2}-\ldots-x-1=0$ is 2^m-1 if $m<n$.

[*B. U. 1922 Pass.* I.]

[Multiply the equation by $x-1$. It then reduces to $x^{n+1}-2x^n+1=0$. Apply § 5·4 to this equation.]

11. Roots of $x^2-2ax+b=0$ and $x^2-2cx+d=0$ represent the vertices of a parallelogram in the Argand's diagram. Prove that

$$(a-c)(a^2-c^2-b+d)=0.$$

12. The roots of the cubic $x^3+ax^2+bx+c=0$ represent the vertices of an equilateral triangle in the Argand's diagram. Prove that $a^2=3b$.

13. The roots of the cubic $x^3+3ax^2+3bx+c=0$ are represented in the Argand's diagram, by three points such that one of them bisects the join of the other two. Prove that $2a^3-3ab+c=0$.

14. In the Argand's digram the roots of the cubic

$$x^3+3px+q=0$$

represent the vertices of a right angled isosceles triangle. Prove that $50p^3=q^2$.

15. If p, q, r be the roots of the cubic $x^3+ax^2+bx+c=0$, prove that $(p+qw+rw^2)^3$ and $(p+qw^2+rw)^3$ are the roots of the quadratic $t^2+(2a^3-9ab+27c)t+(a^2-3b)^3=0$, w being a complex cube root of unity.

If l, m are the cube roots of the roots of this quadratic, such that their product is a^2-3b, the given equation can be thrown into the form $(3x+a)^3-3lm(3x+a)-l^3-m^3=0$.

Solve the equation $3x^3+9x^2+7x+2=0$.

16. In the notation of example 15, prove that $a^2b+b^2c+c^2a$, and $ab^2+bc^2+ca^2$ are the roots of the quadratic,

$$t^2+(ab-3c)t+b^3-2abc+9c^3=0.$$

17. If a, b, c, d are the roots of the equation

$$x^4+px^3+qx^2+rx+s=0$$

prove that $ab+cd$, $ac+bd$, $ad+bc$ are the roots of the cubic

$$f(x)\equiv(px-2r)^2-(x^2-4s)(4x+p^2-4q)=0.$$

[Read § 5·82 and *Example* 4, of the same article]

18. In the same notation, obtain cubics whose roots are

I. $(ab-cd)^2$, $(ac-bd)^2$, $(ad-bc)^2$;

II. $(a+b-c-d)^2$, $(a+c-b-d)^2$, $(a+d-b-c)^2$;

III. $(a-b)^2+(c-d)^2$, $(a-c)^2+(b-d)^2$, $(a-d)^2+(b-c)^2$.

IV. $(a+b)^2+(c+d)^2$, $(a+c)^2+(b+d)^2$, $(a+d)^2+(b+c)^2$.

19. In the same notation, obtain the equation of the sixth degree, whose roots are ab, cd, ac, bd, ad, bc.

[The equations required in the last two examples, are obtained by simple transformation of the cubic of example 17]

20. If the biquadratic equation $x^4+4ax^3+6bx^2+4cx+d=0$ can be put into the form $(x^2+2ax)^2+l(x^2+2ax)+m=0$, prove that $2a^3-3ab+c=0$. Prove further that it can then be thrown into the form $(x^2+2ax+c/a)^2=c^2\div a^2-d$

Solve the equation $x^4+4x^3+12x^2+16x+7=0$.

CHAPTER VI.

ALGEBRAIC EQUATIONS. (PART II.)

6·01 In this chapter, the coefficients of an equation or a polynonial will be supposed to be real rational numbers.

6·1. *If $a+ib$ where a and b are real, is a root of $f(x)=0$, then the conjugate complex number $a-ib$ is also a root.*

Divide $f(x)$ by $(x-a-ib)(x-a+ib)$ or by $(x-a)^2+b^2$. Let the quotient and the remainder be $F(x)$ and $lx+m$ respectively. It is clear that the coefficients in $F(x)$ and $lx+m$ must be all real. We thus have the identity,

$$f(x)=[(x-a)^2+b^2]F(x)+lx+m. \qquad (1)$$

Putting $a+ib$ for x in (1), we have, since $f(a+ib)=0$,

$$0=l(a+ib)+m=la+m+ibl. \qquad (2)$$

From (2), equating the real and imaginary parts, we get

$$la+m=0,\ bl=0;\ \therefore\ l=0,\ (b\neq 0) \text{ and } m=0. \qquad (3)$$

Thus the identity (1) reduces to

$$f(x)=[(x-a)^2+b^2]F(x). \qquad (4)$$

From (4) is is obvious that $f(x)$ vanishes for the value $a-ib$ of x.

Cor. 1. *An equation of an odd degree with real coefficients, has a real root.*

In such an equation, imaginary roots occur in conjugate pairs. Hence their number must be even. Therefore an equation of an odd degree must have at least one real root.

6·11. *If a quadratic surd $a+\sqrt{b}$, where a and b are rational numbers and b is not a square number, is a root of $f(x)=0$, then the conjugate surd $a-\sqrt{b}$ is also a root of $f(x)=0$.*

Divide $f(x)$ by $(x-a-\sqrt{b})(x-a+\sqrt{b})$ or by $(x-a)^2-b$ and denote the quotient and the remainder by $F(x)$ and $lx+m$. It is clear that the coefficients in $F(x)$ and $lx+m$ must be all rational numbers. We thus have the identity,

$$f(x)=[(x-a)^2\mp b]F(x)+lx+m. \qquad (1)$$

Putting $x=a+\sqrt{b}$ in (1), we have, since $f(a+\sqrt{b})=0$,

$$0=l(a+\sqrt{b})+m=la+m+l\sqrt{b}. \qquad (2)$$

$$\therefore \quad la+m=0 \text{ and } l=0; \quad \therefore \quad l=m=0. \qquad (3)$$

Thus the identity (1) reduces to

$$f(x)\equiv[(x-a)^2-b]F(x),$$

from which, it is clear, that $a-\sqrt{b}$ is also a root of $f(x)=0$.

Example 1. Solve the equation

$$F(x)\equiv x^5+2x^4-16x^3-24x^2+15x-2=0,$$

it being known that $2-\sqrt{3}$ is a root.

By §6·11, $2+\sqrt{3}$ must also be a root of $F(x)=0$. Hence $F(x)$ is divisible by $(x-2+\sqrt{3})(x-2-\sqrt{3})$ or x^2-4x+1.

We find by division that the other factor of $F(x)$ is

$$x^3+6x^2+7x-2. \qquad (1)$$

Now we find by inspection that the polynomial (1) vanishes for $x=-2$. $\therefore$ $x+2$ is a factor of (1). The other factor is then found by division to be x^2+4x-1. The roots of the equation are therefore given by $x+2=0$, $x^2-4x+1=0$, and $x^2+4x-1=0$. Hence the roots of $F(x)=0$ are

$$2-\sqrt{3},\ 2+\sqrt{3},\ -2,\ -2+\sqrt{5},\ -2-\sqrt{5}.$$

Example 2. If the coefficients of the equation,

$$x^6+ax^5+bx^4+cx^3+dx^2+ex+f=0$$

are real and the roots are all complex with equal moduli, then $e^3=a^3f^2$ and $de=abf$. [*B. U. 1922 Problems*].

Suppose r is the common modulus. Now the equation whose roots are the roots of the given equation multiplied by r^{-1} is by § 5·63, [121]

$$r^6x^6+ar^5x^5+br^4x^4+cr^3x^3+dr^2x^2+erx+f=0. \qquad (1).$$

Thus the roots of (1) are complex numbers with unity as their common modulus.

Again the coefficients of (1) are real numbers; hence by § 6·1 these complex roots occur in conjugate pairs.

Now conjugate complex numbers with unit modulus are reciprocals of each other. Hence equation (1) is a reciprocal equation (§ 5·622). [119] Therefore we have

$$r^6=f;\ ar^5=er;\ br^4=dr^2 \qquad (2)$$

Hence we easily have $e^3=a^3f^2$ and $de=abf$.

Examples. VI. a.

1. Solve the equation $x^5+x^4+2x^3+2x^2+x+1=0$. It is known that i is a root of the equation.

2. Solve the equation $x^5+2x^4-2x^3-4x^2+9x+18=0$. It is known that $i+\sqrt{2}$ is a root of the equation.

3. Find the equation of least degree with real and rational coefficients, one of its roots being

(1) $\sqrt{2}+\sqrt{3}$; (2) $\sqrt{2}+i\sqrt{3}$ (3) $\sqrt{2}+\sqrt[3]{2}$.

6·2. Commensurable Roots.—*If the leading coefficient of an equation is unity and all other coefficients are integral, prove that it cannot have rational roots that are not integral also.*

Let the equation be

$$f(x)\equiv x^n+p_1x^{n-1}+p_2x^{n-2}+p_3x^{n-3}+\ldots+p_{n-1}x+p_n=0 \quad (1)$$

where $p_1, p_2, \ldots$ are integers. Suppose the equation has a root a/b where a and b are integers prime to each other; then we have, from (1),

$$\frac{a^n}{b^n}+p_1\frac{a^{n-1}}{b^{n-1}}+p_2\frac{a^{n-2}}{b^{n-2}}+p_3\frac{a^{n-3}}{b^{n-3}}+\ldots+p_n=0. \quad (2)$$

Multiplying (2) by b^{n-1}, we get

$$-a^n/b=p_1a^{n-1}+p_2a^{n-2}b+p_3a^{n-3}b^2+\ldots+p_nb^{n-1}. \quad (3)$$

Now the relation (3) is absurd; for, the right hand member of (3) is integral, while the left hand member is a fraction, a^n being prime to b. Hence the rational roots of such an equation must be integral

6·21. *Further, if the equation $f(x)=0$ of § 6·2, has an integral root a, then a is a divisor of the last coefficient p_n.*

Since a is a root of $f(x)=0$, we have

$$a^n+p_1a^{n-1}+p_2a^{n-2}+p_3a^{n-3}+\ldots+p_n=0. \quad (1)$$

(1) may be rewritten thus,

$$p_n=-a(a^{n-1}+p_1a^{n-2}+p_2a^{n-3}+\ldots+p_{n-1}). \quad (2)$$

Now $a^{n-1}+p_1a^{n-2}+\ldots+p_{n-1}$ is an integer. Hence p_n is divisible by a.

6·22. To see whether $f(x)=0$ of § 6·2, has any rational roots, we have only to try the divisors of p_n. To examine any equation for rational roots, we transform it, by § 5·63, into an equation in which the leading or the first coefficient is unity and all other

coefficients are integral, and then we see if any of the divisors of the last coefficient of the transformed equation satisfy it.

Example 1. Examine for rational roots the equation.

$$216x^4+216x^3+72x^2+11x+1=0$$

Dividing the left hand side of the equation by 216, it reduces to

$$x^4+x^3+\tfrac{1}{3}x^2+\tfrac{11}{216}x+\tfrac{1}{216}=0$$

or
$$x^4+x^3+\tfrac{1}{3}x^2+\tfrac{11}{6^3}x+\tfrac{1}{6^3}=0 \qquad (1)$$

The equation whose roots are 6 times those of (1) is by § 5·63

$$x^4+6x^3+12x^2+11x+6=0. \qquad (2)$$

The rational roots of (2) must be integral and they must be divisors of 6. Henee the possible rational roots of (2) are

$$\pm 6,\ \pm 3, \pm 2,\ \pm 1. \qquad (3)$$

Now all coefficients of (2) are positive; hence none of the positive numbers of the set (3) can be a root.

It is easy to see by substitution that -1 is not a root of (2), Again by exhibiting (2) in the form.

$$x^3(x+6)+x(12x+10)+(x+6)=0;$$

it is easy to see that -6 is also not a root of (2). While the remaining numbers -2 and -3 satisfy (2). Hence -1/3 and -1/2 are the roots of the given equation.

Examples VI. b.

It is known that the following equations have commensurable roots. Solve them;

1. $x^5-43x^2+42=0.$
2. $x^5-6x^4-7x^3-x^2+6x+7=0.$
3. $4x^4-21x^2+22x-6=0.$
4. $4x^5-3x^3+3x^2-7x+3=0.$
5. There is no rational number whose square is 7.

6·3. In a numerical equation, written in the normal form, in which the right hand side is zero, if a positive sign follows a negative one, or if a negative sign follows a positive one, we get a *change of sign*. Thus in the equation $x^5-4x^3+4x^2+5x-1=0$, there are three changes of sign. The following theorem is known as **Descarte's Rule of Signs.**

6·31. *The number of positive roots of an equation cannot exceed the number of changes of sign in it.*

We shall first shew that if a polynomial is multiplied by a

factor $x-a$ where a is positive, at least one more change of sign is introduced in the distributed product. For the sake of simplicity, we shall prove it in the case of a particular polynomial; the argument will, however, be found perfectly general.

Let the succession of signs in the given polynomial be

$$+ \quad + \quad + \quad - \quad + \quad + \quad - \quad - \quad + \quad - \quad .$$

It contains five changes of sign.

Writing only the signs that occur in the process of multiplication, we have

$$\begin{array}{llllllllll}
+ & + & + & - & + & + & - & - & + & - \qquad \text{1st row,} \\
+ & - & & & & & & & & \\
\hline
+ & + & + & - & + & + & - & - & + & - \qquad \text{2nd row.} \\
 & [- & - & - & + & - & - & + & + & -] + \qquad \text{3rd row.} \\
\hline
+ & \ldots & \ldots & - & + & \ldots & - & \ldots & + & - \quad + \qquad \text{4th row.}
\end{array}$$

We can write the whole of the 2nd row of signs; on the other hand, the polynomial may be incomplete, and hence except the last sign, the whole of the 3rd row cannot be written down. However, we are sure of some of the signs in the 4th row. The 1st sign in the 4th row is obviously positive. Now either there is no term which can be written under the second sign of the 2nd row, or there is a term with a negative coefficient arising from the multiplication of two terms with opposite signs. In either case, the corresponding sign in the 4th row is doubtful. The sign in the 4th row corresponding to the 3rd sign in the 2nd row is similarly indeterminate. Either there is no term which can be written under the 4th sign in the 2nd row or there is a term with a negative sign arising as a product of two terms with contrary signs; in either case the corresponding sign in the 4th row is negative. The signs in the 4th row corresponding to the 5th and 7th signs in the 2nd row are similarly + and −; those, corresponding to the 6th and 8th are doubtful, while those corresponding to the last three are +, −. +.

Thus whenever a positive sign follows a negative one in the 1st row, that positive sign persists in the 4th row; similarly if a negative sign follows a positive one, that negative sign persists in the 4th row. On the other hand, if there is a continuity of two like signs in the 1st row, the corresponding sign in the 4th row is doubtful.

Further, the last sign in the 4th row is opposite to the last sign in the 1st row. It follows, as a consequence, from the above argument, that all the signs in the 4th row, which are not doubtful, must be alternately positive and negative or negative and positive. Now we observe that in a succession of signs with the two extreme signs opposite, there must be an odd number of changes of sign. Hence, we conclude that the number of changes of sign in the final product must be greater than that in the original polynomial by an odd number.

Now imagine that the product of all factors corresponding to negative and imaginary roots is multiplied out. Every time the resulting polynomial is multiplied by a factor of the form $x-a$ corresponding to a positive root, an additional odd number of changes of sign are introduced. Hence, there cannot be more positive roots than there are changes of sign in the equation.

To find an upper limit to the number of negative roots of an equation $f(x)=0$, we apply the above theorem to the equation $f(-x)=0$ whose roots, we know, are equal in magnitude, but opposite in sign, to those of $f(x)=0$, (§ 5·61).

The product of factors corresponding to negative roots and pairs of conjugate complex roots is of the form,

$$\text{II}\ (ax+b)\ \text{II}\ [(x-l)^2+m^2]$$

where a and b are of the same sign. It is clear that if this product is distributed the two extreme signs in the resulting polynomial must be alike. Now in a series of signs in which the extreme signs are alike, there is an even number (including zero) of changes of sign. It follows that in such a polynomial the number of changes of sign must be even. Hence the above rule may be enunciated more precisely thus:

The number of changes of sign in $f(x)$ either equals, or exceeds by an even number, the number of positive roots of $f(x)=0$

The theorem gives only an upper limit to the number of real roots of an equation; hence its use is very much limited.

Example 1. Prove that the equation $x^7-x+2=0$, has at least 4 imaginary roots.

The equation has two changes of sign. Hence it cannot have more than two positive roots; while the equation $(-x)^7-(-x)+2=0$, or $x^7+x-2=0$, has only one positive root. Therefore the given equation cannot have more than one negative root (§ 6·31). Now the equation is of the 7th degree, hence it must have at least 4 complex roots.

Example 2. Construct the equation whose roots are the squares of the roots of $x^5+x^3+x^2+2x+3=0$ and hence or otherwise prove that the given equation has 4 complex roots.

[*B. U. 1917 Hons*].

Suppose a, b, c, d, e are the roots of the equation; then, denoting $x^5+x^3+x^2+2x+3$ by $f(x)$, we have

$$f(x)\equiv x^5+x^3+x^2+2x+3\equiv(x-a)(x-b)(x-c)(x-d)(x-e). \quad (1)$$

Writing $(-x)$ for x in the identity (1), we have

$$f(-x)=(-x-a)(-x-b)(-x-c)(-x-d)(-x-e); \quad (2)$$

$$\therefore\ f(x)\ f(-x)=(x^2+3)^2-x^2(x^4+x^2+2)^2$$

$$=-(x^2-a^2)(x^2-b^2)(x^2-c^2)(x^2-d^2)(x^2-e^2). \quad (3)$$

Writing y for x^2 in (3), we have the identity

$$(y+3)^2-y(y^2+y+2)^2\equiv-(y^5+2y^4+5y^3+3y^2-2y-9)$$

$$=-(y-a^2)(y-b^2)(y-c^2)(y-d^2)(y-e^2). \quad (4)$$

Hence, the equation whose roots are a^2, b^2, c^2, d^2, e^2 is

$$y^5+2y^4+5y^3+3y^2-2y-9=0. \quad (5)$$

Now this equation has only one change of sign. Hence, it has at most one positive root. It has therefore four negative or complex roots at least. Now the square roots of complex and negative quantities are complex. Hence the given equation must have at least four imaginary roots. Since the equation is of odd degree, the remaining fifth root must be real (§ 6·1, cor. 1). Hence the equation has four complex roots.

Examples. VI. c.

1. Prove that $x^{2n}-1=0$ has only two real roots, 1 and -1.
2. Prove that $x^{2n-1}-1=0$ has only one real root, 1.
3. Prove that $x^{2n}+1=0$ has no real root.
4. Prove that $x^{2n+1}+1=0$ has only one real root -1.
5. Prove that $x^{2n-1}-p^2x+q^2=0$ has at least $2n-4$ complex roots.

6. Prove that the equation $x^{2n-1}+4x-1=0$ has one and only one real root.

7. Discuss the nature of the roots of $x^{2n}+ax^n+b=0$, a, b being real numbers.

6·4. *As $|x|$ tends to zero, the polynomial of the form $p_0x^n+p_1x^{n-1}+p_2x^{n-2}+p_3x^{n-3}+\ldots+p_{n-1}x$ in which the constant term is absent, also tends to zero.*

By § 3·713, $|p_0x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_{n-1}x|$

$$\begin{aligned} &> |p_0x^n|+|p_1x^{n-1}|+|p_2x^{n-2}|+\ldots+|p_{n-1}x| \\ &= |p_0|\,|x|^n+|p_1|\,|x|^{n-1}+|p_2|\,|x|^{n-2}+\ldots \\ &\qquad +|p_{n-1}|\,|x|. \quad (\S\ 3{\cdot}8) \\ &< A\,(R^n+R^{n-1}+\ldots\ldots+R^2+R) \end{aligned}$$

where $R=|x|$ and A is not less than any of the quantities

$$|p_0|, |p_1|, |p_2|, \ldots, |p_{n-1}|,$$

$$\therefore |p_0x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_{n-1}| < \frac{AR(1-R^n)}{1-R} < \frac{AR}{1-R} < h$$

where $R<h/(A+h)$ and h is an arbitrarily small positive quantity. Thus we prove the theorem.

The above proof is applicable to the case where x and p's represent complex quantities also.

Cor. 1. *The sign of a polynomial*

$$f(x)\equiv p_0x^n+p_1x^{n-1}+\ldots\quad+p_{n-1}x+p_n,$$

for sufficiently small values of $|x|$, is governed by the constant term p_n.

For sufficiently small values of $|x|$,

$$|p_0x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_{n-1}x| < |p_n|. \quad (\S\ 6{\cdot}4).$$

For all such values of x the sign of $f(x)$ is clearly the same as that of p_n.

Cor. 2. *The sign of a polynomial*

$$f(x)\equiv p_0x^n+p_1x^{n-1}+\ldots+p_n,$$

for sufficiently large values of $|x|$ is governed by the sign of p_0x^n.

We have

$$f(x)\equiv p_0x^n\left(1+\frac{p_1}{p_0}\frac{1}{x}+\frac{p_2}{p_0}\frac{1}{x^2}+\frac{p_3}{p_0}\frac{1}{x^3}+\ldots+\frac{p_n}{p_0}\frac{1}{x^n}\right)$$

Now as $|x|$ increases, $1\div|x|$ or $|1/x|$ decreases.

Therefore by cor. 1, above, for sufficiently large values of $|x|$, the sign of

$$1+\frac{p_1}{p_0}\frac{1}{x}+\frac{p_2}{p_0}\frac{1}{x^2}+\frac{p_3}{p_0}\frac{1}{x^3}+\ldots+\frac{p_n}{p_0}\frac{1}{x^n}$$

is the same as the sign of 1, and so, it is positive.

Hence for sufficiently large values of $|x|$, the sign of the polynomial is the same as that of p_0x^n.

Cor. 3. *If $f(a)\neq 0$, then $f(a-h)$, $f(a)$ and $f(a+h)$ have the same sign, h being a sufficiently small positive number.*

For by § 5·3, we have

$$f(a-h)=f(a)-\frac{h}{1!}f'(a)+\frac{h^2}{2!}f''(a)-\ldots,$$

and $$f(a+h)=f(a)+\frac{h}{1!}f'(a)+\frac{h^2}{2!}f''(a)+\ldots.$$

Now by cor. 1. above, the signs of the two right hand polynomials in h, are the same as that of $f(a)$.

Cor. 4. *On the other hand, if $f(a)=0$, and $f'(a)\neq 0$, $f(a-h)$ and $f'(a)$ have opposite signs and $f(a+h)$ and $f'(a)$ have like signs.*

The results are obvious from the two expansions

$$f(a-h)=-hf'(a)+\frac{h^2}{2!}f''(a)-\ldots$$

and $$f(a+h)=hf'(a)+\frac{h^2}{2!}f''(a)+\ldots\ldots$$

Cor. 5. *Any polynomial $f(x)$ is a continuous function of x.*

We have, by § 5·3,

$$f(x+h)-f(x)=\frac{h}{1!}f'(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3!}f'''(x)+\ldots\ldots.$$

Now for a particular value of x, the right hand expansion is a polynomial in h. Hence its numerical value tends to zero as $|h|$ tends to zero. Thus we see that the absolute value of the change in the polynomial, $f(x+h)-f(x)$ due to a small change, $|h|$ in the argument x, can be made as small as we please. Hence $f(x)$ is a continuous function of x.

6·5. *If $f(a)$ and $f(b)$ are of opposite signs, at least one real root of $f(x)=0$ must lie between a and b.*

As x varies from a to b, $f(x)$ varies continuously from $f(a)$ to

$f(b)$ (§ 6·4, cor. 5.) assuming, at least once, every value between $f(a)$ and $f(b)$. Since $f(a)$ and $f(b)$ are of opposite signs, zero lies between them; hence $f(x)$ must be zero, at least once, for a value of x between a and b.

For a more general and rigorous proof of this theorem, read Hardy's *Pure Mathematics*, § 100.

Cor. 1. If the degree of $f(x)=0$ is odd, the equation has at least one real root.

By § 6·4 cor. 2. for sufficiently large values of $|x|$, the sign of $f(x)$ is identical with the sign of the first term. Now the first term is an odd power of x. Hence its sign for a numerically large negative value of x must be different from that for a numerically large positive value. Therefore by the above theorem, at least one root of the equation $f(x)=0$ must be between these values of x. (*c. f* § 6·1, cor. 1.)

Cor. 2. *If the degree of* $f(x)=0$ *is even and the leading coefficient and the constant term have opposite signs the equation has at least two real roots*

By § 6·4, cor. 2, it is easy to see that either $f(-\infty)$ and $f(\infty)$ are both positive and $f(0)$ is then negative; or that $f(-\infty)$, $f(\infty)$ are both negative and $f(0)$ is positive. In either case, the equation has at least one negative root and one positive root.

Example 1. Show that the roots of the equation $x^3-12x-10=0$ are all real and locate them between pairs of consecutive integers.
[*B. U. 1914 Pass* I.]

Denote $x^3-12x-10$ by $f(x)$. Then by putting $f(x)$ into the form $f(x)\equiv x(x^2-15)+(3x-10)$, it is easily seen that for values of x equal to and greater then 4, $f(x)$ is positive. Hence the equation has no root greater than 4. Substituting successively values 4, 3, 2, 1. 0, −1, −2,... for x in the equation, we have

$f(4)=6$; $f(3)=-19$; $f(2)=-26$; $f(1)=-21$ $f(0)=-10$; $f(-1)=1$

$f(-2)=6$; $f(-3)=-1$.

Thus, roots of the equation lie between −3 and −2, −1 and 0, and 3 and 4.

6·51. *If $f(\alpha)$ and $f(\beta)$ have opposite signs, then an odd number of roots of $f(x)=0$ lie between α and β.*

Let $a, b, c, \ldots k, l$, be the roots lying between α and β, Suppose further that $\alpha < a < b < c \ldots k < l < \beta$ then we have the identity

$$f(x) \equiv (x-a)(x-b)\ldots(x-l)\ F(x) \qquad (1)$$

where $F(x)$ is the product of factors corresponding to the pairs of conjugate complex roots (§ 6·1) and to the real roots that lie outside the range between α and β.

Now putting α and β in (1), we have

$$f(\alpha) = (\alpha-a)(\alpha-b)(\alpha-c)\ldots(\alpha-l)\ F(\alpha); \qquad (2)$$

$$f(\beta) = (\beta-a)(\beta-b)\ \beta-c)\ldots(\beta-l)\ F\beta). \qquad (3)$$

Since $f(\alpha)$ and $f(\beta)$ have opposite signs and $F(\alpha)$, $F(\beta)$, like signs,* it follows that the products

$$(\alpha-a)(\alpha-b)(\alpha-c)\ldots(\alpha-k)(\alpha-l), \qquad (4)$$

and $$(\beta-a)(\beta-b)(\beta-c)\ldots(\beta-k)(\beta-l)$$

have opposite signs. Now $\beta-a$, $\beta-b$, $\beta-c, \ldots \beta-l$ are all positive, while $\alpha-a$, $\alpha-b, \ldots(\alpha-l)$ are all negative. Hence the number of factors in (4) and therefore the number of quanties $a, b, \ldots$ must be odd.

We may similarly prove that, if $f(\alpha)$ and $f(\beta)$ have the same sign, an even number of roots lie between α and β.

All these results can be verified intuitively by an appeal to the graph of the function $f(x)$.

* *N. B.* Now pairs of conjugate complex roots give rise to factors of the form $A[(x-l)^2+m^2]$. Such factors clearly retain the same sign for all real values of x. Similarly a factor of the form $b(x-p)$ where p does not lie between α an β, has the same sign for values α and β of x. Hence the sign of $F(x)$ is the same for values α and β of x.

Example 1. Prove that the roots of the cubic

$$(x-1)(x-5)(x-3)+2(x-2)(x-4)(x-6)=0$$

are all real

Denote the left hand member of the equation by $f(x)$; then we easily have,

$f(1) = -30$; $f(2) = 3$, $f(3) = 6$, $f(4) = -3$, $f(5) = -6$ and $f(6) = 15$.

Hence 3 of the roots are real. Since the equation is a cubic it cannot have more than three roots. Hence the result.

6·6. *Between every two real roots of $f(x)$, there is at least one real root of $f'(x) = 0$.*

Let a and b be two consecutive roots of the equation $f(x) = 0$, repeated p and q times, and let $\psi(x)$ be the product of factors corresponding to the remaining roots. We have thus the identity

$$f(x) \equiv (x-a)^p (x-b)^q \psi(x). \qquad (1)$$

Differentiating this identity (§ 5·31), we have another identity

$$f'(x) \equiv (x-a)^{p-1}(x-b)^{q-1}[p(x-b) + q(x-a)\psi(x) + (x-a)(x-b)\psi'(x)]$$

$$= (x-a)^{p-1}(x-b)^{q-1}F(x) \text{ say}. \qquad (2)$$

Now $F(a) = p(a-b)\psi(a)$ and $F(b) = q(b-a)\psi(b)$. (3)

Since $p\psi(a)$ and $q\psi(b)$ have like signs (See N. B. § 6·51), from (3), we conclude that $F(a)$ and $F(b)$ have opposite signs. Hence by § 6·51, $F(x)$ and therefore $f(x)$ must vanish for an odd number of values of x lying between a and b, Hence the theorem is proved.

A simple verification of the theorem can be obtained by referring to the graph of $f(x)$. This theorem is attributed to Rolle. For a more general result applicable to all continuous functions, read Hardy's '*Pure Mathematics*' § 121, Theorem B.

Cor. 1. *If all the roots of $f(x) = 0$ are real, then those of $f'(x) = 0$ are also all real and they separate those of $f(x) = 0$.*

Since all the n roots of $f(x) = 0$, are real $n-1$ of those of $f'(x) = 0$ must be real. Now $f'(x) = 0$ being of the $n-1$ th dogree, can not have more than $n-1$ roots. Hence all the roots of $f'(x) = 0$ are real. That the roots of $f'(x) = 0$ separate those of $f(x) = 0$ follows from the fact that between every two roots of $f(x) = 0$ there exists at least one root of $f'(x) = 0$. Applying the same argument to $f'(x) = 0$, we find that all the roots of $f''(x) = 0$ must be real and must separate those of $f'(x) = 0$. Repeating the argument, we have a series of equations $f(x) = 0$, $f'(x) = 0$, $f''(x) = 0$...such that each of

them has all its roots real and separates those of the equation that immediately precedes it.

Cor. 2. *If $f'(x)=0$ has r real roots, then $f(x)=0$ can not have more than $r+1$ real roots.*

For if $f(x)=0$ have more than $r+1$ real roots, $f'(x)=0$ would have more than r real roots which is inconsistent with the hypothesis.

Example 1. Determine for what range of numerical values of k, the roots of $2x^3-9x^2+12x-k=0$ are all real.

[*B. U. 1924 Pass* I.]

Denote $2x^3-9x^2+12x-k$ by $f(x)$; then the roots of

$$f'(x)\equiv 6x^2-18x+12=0 \text{ are } 2 \text{ and } 1.$$

Now $f(-\infty)$, $f(1)$, $f(2)$ and (∞), where ∞ represents as usual a sufficiently large number, are respectively (§ 6·4, cor. 2.)

$$-ve,\ 5-k,\ 4-k,\ +ve. \qquad (1)$$

Since the roots of $f(x)=0$ are all real, and those of $f'(x)=0$ separate them, § 6·5 requires that the signs of quantities (1) should alternately change.

$\therefore\ 5-k>0$ and $4-k<0$ $\therefore$ k must lie between 4 and 5. When k is either 4 or 5, the given equation, as we shall show later on (§ 6·7), has a pair of equal roots and so all its roots are real in this case also.

Example 2. Show that all the roots of the following equation are real

$$1-\frac{n}{1!}\frac{n+1}{1!}x+\frac{n(n-1)}{2!}\frac{(n+2)(n+1)}{2!}x^2-\ldots$$
$$+(-)^n\frac{(2n)(2n-1)\ldots(n+2)(n+1)}{n!}x^n=0$$

[*B. U. 1916 Hons*]

Denoting the left hand member of the equation by $f(x)$, it is easy to prove the identity

$$\frac{1}{n!}\frac{d^n}{dx^n}x^n(1-x)^n\equiv f(x).$$

Now all the $2n$ roots of the equation $x^n(1-x)^n=0$ are real and equal to either zero or one. Hence by §6.6, cor. 1. all the roots of $f(x)=0$ are real and lie between zero and one.

Example 3. Show that the equation

$$f_n(x) \equiv 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} = 0$$

has one real root if n is odd and none if n is even.

$f_1(x) = 0$ has one real root -1.

$f_2(x) = 0$ has obviously no real root.

$f_3(x) = 0$ is an equation of odd degree ; so it must have at least one real root (§6·5, cor. 1). Further $f_3'(x) \equiv f_2(x) = 0$, has no real root. Hence by §6.6, cor. 2, $f_3(x) = 0$, cannot have more than one real root. So $f_3(x) = 0$ has one and only one real root, c say.

Now $f'_4(x) \equiv f_3(x) = 0$ has only one real root c. Therefore $f_4(x) = 0$ can not have more than two real roots. Again, by § 6·4, cor. 2, $f(-\infty)$ is positive; $f_4(c) \equiv f_3(c) + \frac{c^4}{4!} = \frac{c^4}{4!}$ = a positive quantity; also $f(\infty)$ is positive by §6.4 cor. 2. Hence $f_4(x) = 0$ has no real root. In this way it is easy to establish the theorem for all values of x.

This example is taken from Hardy's *Pure Mathematics*, page 398, example 39, where a proof on similar lines is suggested.

Example 4. Prove that if $f(x)$ is a polynomial in x all of whose roots $a, b, c, \ldots l$ are real and distinct, then the roots of

$$F(x) \equiv f(x) \left\{ \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c} + \ldots + \frac{L}{x-l} \right\} = 0$$

where $A, B, C, \ldots, L$ are constants all of which have the same sign, are also all real and distinct and separate those of $f(x) = 0$ and conversely that any polynomial $F(x)$ whose degree is less by one than that of $f(x)$ and whose roots are real and distinct and separate those of $f(x) = 0$ can be expressed in this form in one and only one way. [*B. U. 1923 Hons I.*]

Suppose $a, b, c, \ldots, l$, are arranged in the descending order of magnitude, then by §5.31 we have

$$f'(a) = k(a-b)(a-c)\ldots(a-l),$$
$$f'(b) = k(b-a)(b-c)\ldots(b-l).$$
$$f'(c) = k(c-a)(c-b)\ldots(c-l),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

where k is some constant. We see at once from these equations that $f'(a)$, $f'(b)$, $f'(c)\ldots f(l)$, are alternately positive and negative or negative and positive. Now

$$F(a)=Af'(a);\ F(b)=Bf'(b);\ F(c)=Cf'(c)\ldots\ldots$$

and A, B, C,...are of the same sign; hence $F(a)$, $F(b)$, $F(c)\ldots$ are alternately positive and negative. Therefore the roots of the equation $F(x)=0$ lie between a and b; b and c; c and d; and so on.

To prove the converse theorem, we observe that $F(x)$ can be put into the form,

$$f(x)\left\{\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}+\ldots+\frac{L}{x-l}\right\}.$$

The assumption is legitimate; for the number of quantities A, B, ...is equal to the number of terms in $F(x)$, and so by equating the coefficients of like powers of x, we get sufficient number of linear equations to determine them uniquely. Now by §5·3, $F(a)$, $F(b)$, $F(c)$,...are equal to $Af'(a)$, $Bf'(b)$,...respectively.

Since a, b, c,... are the roots of $f(x)=0$ arranged in descending order of magnitude, $f'(a)$, $f'(b)$, $f'(c)\ldots$ are alternately positive and negative. Again the roots of $F(x)=0$ separate those of $f(x)=0$; therefore, $F(a)$, $F(b)$,... *i. e.* $Af'(a)$, $Bf'(b)$,... are alternately positive and negative. Hence A, B, C,... must be of the same sign.

Examples VI. d.

Prove that the roots of the following equations are all real:

1. $x^3-3x-1=0$. 2. $x^3-12x-15=0$. 3. $4x^3-6x^2+1=0$.

4. $x^4+4x^3-2x^2-12x+7=0$. 5. $2x^4-8x^3+8x^2-1=0$.

6. If all the roots of the equation $2x^3+3x^2-36x+c=0$ are real, prove that $-81<c<44$.

7. Determine graphically or otherwise for what values of l the equation $2x^3-15x^2+24x-l=0$ has three real roots. [*B. U. 1921 Hons. 1.*]

8 Prove that the equation $(x-a)^3+3b^2x+c=0$ has only one real root, a, b, c being real.

9. Prove that the equation $4x^5-5x+c=0$ can not have more than three real roots and that when it has three real roots $c^2-1>0$.

10. The roots of the cubic $x^3+3bx+c=0$ are all real if

$$c^2+4b^3>0.$$

11. The roots of the cubic $2x^3+3ax^2+c=0$ are all real if

$$c(c+a^3)>0.$$

12. Show that the equation $x^5+5ax^3+b=0$ has one and only one real root either if a is positive or if a is negative and $b^2+108a^5>0$. [*B. A. Hons 1923.*]

6·7. Equal Roots. *If a is a root (not necessarily real) of $f(x)=0$ repeated r times, then it is also a root of $f'(x)=0$ repeated $r-1$ times.*

Since a is a repeated root of the rth order, $f(x)$ can be expressed in the form

$$f(x)=(x-a)^r F(x), \qquad (1)$$

where $F(x)$ is the product of factors corresponding to the remaining roots of the equation. Differentiating (1), we have, by § 5·31,

$$f'(x)=(x-a)^{r-1}\,[rF(x)+(x-a)\,F'(x)]. \qquad (2)$$

Since $rF(x)+(x-a)F'(x)$ is clearly a polynomial, it is obvious from (2) that $f'(x)$ is divisible by $(x-a)^{r-1}$. a is therefore a root of $f'(x)=0$ repeated $r-1$ times.

6·71. Applying this theorem to $f'(x)=0$, we find that a must be a root of $f''(x)=0$ repeated $r-2$ times. Similarly a is a root of $f'''(x)=0$ repeated $r-3$ times and so on.

6·72. *Conversely, if $f(a)=f'(a)=f''(a)=\ldots=f^{(r-1)}(a)=0$, then a is a root of $f(x)=0$ repeated r times.*

By writing $x-h$ for x and a for x in the identity of §5·3, we have

$$f(a+x-a)=f(x)=f(a)+\frac{(x-a)}{1!}f'(a)+\frac{(x-a)^2}{2!}f''(a)+\ldots$$

$$+\frac{(x-a)^r}{r!}f^{(r)}(a)+\ldots \qquad (1)$$

Now $f(a)=f'(a)=\ldots=0$

$\therefore$ (1) reduces to $f(x)=(x-a)^r F(x)$, where $F(x)$ is a polynomial. Hence a is a repeated root of $f(x)=0$ of the rth order.

Example 1. One root of $x^5-39x^3+104x^2-28x-80=0$ is $-4+\sqrt{11}$ and two others are equal to one another. Solve the equation [*B. U. 1920 Hons.* 1]

By § 6·11, $-4-\sqrt{11}$ is also a root of the equation. Therefore $(x+4-\sqrt{11})(x+4+\sqrt{11})$ or x^2+8x+5 is a factor of

$$x^5-39x^3+104x^2-28x-80.$$

By division, the other factor is found to be $x^3-8x^2+20x-16$.

By hypothesis $x^3-8x^2+20x-16$ has a repeated factor, therefore it must be a factor of its derived function $3x^2-16x+20$, the factors of which are $x-2$, $3x-10$. [By substitution, 2 is found to be a root of $x^3-8x^2+20x-16=0$, and therefore it must be its repeated root.] By considering the sum of the roots of this cubic, the 3rd. root is found to be 4. Thus the roots of the given equation are

$$2,\ 2,\ 4,\ -4-\sqrt{11},\ -4+\sqrt{11}.$$

Example 2. Factorise the polynomial

$$f(x)\equiv x^7-x^6-2x^4+2x^3+x-1$$

without solving the equation $f(x)=0$. It is known that the equation has repeated roots.

By § 6·7, $f'(x)=0$ and $f(x)=0$ must have common roots; therefore $f(x)$ and $f'(x)$ must have common factors.

Now the *H. C. F.* of $f(x)$ and $f'(x)$ is x^4-x^3-x+1. Denote it by $f_1(x)$. If $f_1(x)$ has any repeated factors, they must be common to $f_1(x)$ and $f_1'(x)$. The *H. C. F* of these two polynomials is $x-1$. Therefore $(x-1)^2$ is a factor of $f_1(x)$. The other factor of $f_1(x)$ is easily found to be x^2+x+1

$$\therefore\ f_1(x)\equiv(x-1)^2(x^2+x+1).$$

Now $f_1(x)$ is a common factor of $f(x)$ and $f'(x)$. Therefore $(x-1)^3(x^2+x+1)^2$ must be a factor of $f(x)$.

Since $f(x)$ is of the 7th degree it cannot have any more factors; hence

$$f(x)=(x-1)^3(x^2+x+1)^2.$$

In this way it is possible to throw any polynomial $f(x)$ into the form

$$F_1(x)[F_2(x)]^2.[F_3(x)]^3\ldots$$

by means of elementary algebraical processes only.

(*c. f. B. U. 1919 Hons. I*].

Example 3. Show that, if $ax^3+3bx^2+3cx+d=0$ has two equal roots, they are each equal to $\frac{bc-ad}{ac-b^2}=2\frac{bd-c^2}{bc-ad}$. [*B. U. 1924 Pass I*]

The factor corresponding to the repeated root must be common to $ax^3+3bx^2+3cx+d$ and its derived function $3ax^2+6bx+3c$. Hence it is a factor of

$$ax^3+3bx^2+3cx+d-x(ax^2+2bx+c) \text{ i. e. of } bx^2+2cx+d.$$

The repeated factor is thus the common factor of $ax^2+2bx+c$ and $bx^2+2cx+d$. It is therefore a factor of

$$a^{-1}(ax^2+2bx+c)-b^{-1}(bx^2+2cx+d) \text{ i.e. of } 2x(ba^{-1}-cb^{-1})+ca^{-1}-b^{-1}d$$

Hence the repeated factor of $ax^3+3bx^2+3cx+d$ is given by

$$2x(b^2-ac)+(cb-ad).$$

∴ One of the equal roots of the given equation is

$$(bc-ad)/2(ac-b^2).$$

Example 4. Prove that there are three values of c for which the equation $ax^3+3bx^2+3cx+d=0$ has equal roots; if $b^3=a^2d$; two of these values coincide and the third is $-\frac{5}{4}$ of either.

[*B. U. 1921, Hons.* I]

From *example* 3 above, the repeated root is given by

$$x=(bc-ad)\div 2(ac-b^2)$$

Eliminating x, the necessary condition for repeated roots is

$$4\,(bd-c^2)(ac-b^2)=(bc-ad)^2. \qquad (1)$$

Now (1) is a cubic in c; hence the given equation has equal roots for three values of c. Putting $d=b^3/a^2$ in (1), we have

$$4(b^4-a^2c^2)(ac-b^2)=(abc-b^3)^2. \qquad (2)$$

Two of the roots of this cubic in c are each obviously equal to b^2/a; the remaining is then seen to be $-5b^2/4a$.

Examples. VI. e.

The following equations are known to have equal roots; solve them:

(1) $x^4-6x^2+8x-3,=0.$

(2) $x^4-11x^2+12x+4=0.$

(3) $4x^5-15x^3-5x^2+15x+9=0.$

(4) $x^5-2x^3+4x^2+8x-21=0.$

(5) $x^5-8x^3-24x^2-28x-16=0.$

(6) The necessary and sufficient condition that the cubic $x^3+3px+r=0$ should have a pair of equal roots is $r^2+4p^3=0$.

(7) If the equation $x^3+3ax^2+3bx+c=0$ has a pair of equal roots, prove that the ratio of either of them to the remaining third root is $(ac-bd)^3 \div c(2b^2-2ab)^3$.

(8) If the equation $(n-1)x^n+npx^{n-1}+q=0$ has two equal roots other than zero, prove that $q=(-p)^n$ and that the equation giving the remaining $n-2$ roots is

$$(n-1)x^{n-2}-(n-2)px^{n-3}+(n-3)p^2x^{n-4}-(n-4)p^3x^{n-5}+\ldots=0.$$

(9) Prove that if m and n are positive integers,

$$mx^{m+n}-(m+n)x^m+n$$

is divisible by $(x-1)^2$ and that the quotient is

$$mx^{m+n-2}+\ldots+p_rx^r+\ldots+n$$

where $p_r=m(m+n-r-1)$ if $r<m$ and $p_r=(r+1)n$ if $r<m$.

[*B. U. 1923 Problems.*]

6·8. Sturm's Theorem.

Assume that the equation $f(x)=0$ has no repeated roots. Now apply the process of calculating the highest common factor to $f(x)$ and its derived function $f_1(x)$ with this modification of method that the sign of every remainder is to be changed before it is used as the next divisor. If the remainders with their signs so changed, are $f_2(x), f_3(x), f_4(x) \ldots f_n(x)$, then these functions together with $f(x)$ and $f_1(x)$ are called *Sturm's functions*.

6·81. Suppose $q_1, q_2, q_3, \ldots q_{n-1}$ are the quotients that occur in the process of calculating Sturm's functions; then we have the identities

$$f(x)=q_1f_1(x)-f_2(x), \qquad (1)$$

$$f_1(x)=q_2f_2(x)-f_3(x), \qquad (2)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$f_{r-1}(x)=q_rf_r(x)-f_{r+1}(x), \qquad (r)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$f_{n-3}(x)=q_{n-2}f_{n-2}(x)-f_{n-1}(x),$$

$$f_{n-2}(x)=q_{n-1}f_{n-1}(x)-f_n(x). \qquad (n-1)$$

1st *property*:- Since $f(x)=0$ has no repeated roots, $f(x)$ and $f_1(x)$ have no common factor (§6·7); therefore from the identity

(1), we see that $f_1(x)$ and $f_2(x)$ have no common factor. In the same way from the identity (2), we see that $f_2(x)$ and $f_3(x)$ have no common factor. Thus by repeated argument, we prove that *no two consecutive Sturm's functions vanish for the same value of* x.

2nd property:—We have seen that $f_{n-2}(x)$ and $f_{n-1}(x)$ have no common factor; therefore $f_n(x)$ *is a constant different from zero.*

3rd property:—If a real number a is a root of one of the Sturm's functions $f_r(x)$ say, then for this value a of x, the two neighbouring functions $f_{r-1}(x)$ and $f_{r+1}(x)$ are equal in magnitude and opposite in sign; For putting a in the identity

$$f_{r-1}(x) = q_r f_r(x) - f_{r+1}(x),$$

we have

$$f_{r-1}(a) = -f_{r+1}(a)$$

This relation proves the required property.

6.82. We are now in a position to prove the important theorem attributed to Sturm. It enables us to determine the exact number of real roots of an equation lying between any two given values of x.

The number of real roots of $f(x)=0$ lying between a and b is equal to the difference between the number of changes of sign (§ 6·3) in the series of signs of $f(a)$, $f_1(a)$, $f_2(a),\ldots f_n(a)$ and the number of changes of sign in the series of signs of $f(b)$, $f_1(b)$, $f_2(b),\ldots f_n(b)$, where $f(x)$, $f_1(x)$, $f_2(x)\ldots f_n(x)$ are the Sturm's functions of the equation $f(x)=0$. The equation $f(x)=0$ has, of course, no repeated roots.

We shall examine how the series of signs of the Sturm's functions is affected, as we successively substitute values of the real variable x in the neighbourhood of some number c.

Case I. If c is not a root of any of the equations

$$f(x)=0,\ f_1(x)=0,\ f_2(x)=0,\ldots$$

then by §6.4, cor. 3, the series of signs of

$$f(c-h),\ f_1(c-h),\ f_2(c-h),\ldots f_{n-1}(c-h),\ f_n(c-h),$$

is identical with the series of signs of

$$f(c+h),\ f_1(c+h),\ f_2(c+h),\ldots f_{n-1}(c+h),\ f_n(c+h),$$

h being supposed to be a small positive number. Hence in this

case, the numbers of changes of signs before and after we pass through c, are the same.

Case II. Now suppose c is a root of one of the intermediate equations $f_r(x)=0$, but not of the given equation $f(x)=0$.

We have $f_r(c)=0$; therefore $f_{r-1}(c)$ and $f_{r+1}(c)$ are not zero (1st proparty § 6·81). Hence by § 6·4, cor. 3, $f_{r-1}(c-h)$, $f_{r-1}(c)$ and $f_{r-1}(c+h)$ have the same sign and $f_{r+1}(c-h)$, $f_{r+1}(c)$, $f_{r+1}(c-h)$ also have the same sign.

Now since $f_r(c)=0$, $f_{r-1}(c)$ and $f_{r+1}(c)$ have opposite signs (2nd property, § 6·81); therefore in the arrangement of the signs of $f_{r-1}(x)$, $f_r(x)$ and $f_{r+1}(x)$ before and after we pass through c the extreme signs are opposite; so there is one change of sign before we pass through c and one and only one change of sign after we pass through c. Thus in this case also, although the order of signs may be disturbed, the total number of changes of sign is not affected.

Even if c $[f(c)\neq 0]$ be a root of two or more intermediate equations like $f_r(x)=0$, since it cannot be a root of two consecutive equations (1st property), the number of changes of sign is not affected.

Case III. Now suppose c is a root of the given equation $f(x)=0$. Then by § 6·4, cor. 4, $f(c-h)$ and $f_1(c)$ have opposite signs and $f(c+h)$ and $f_1(c)$ have like signs; again since $f_1(c)\neq 0$, by § 6·4, cor. 3, $f_1(c-h)$ $f_1(c)$ and $f_1(c+h)$ have the same sign. Hence, $f(c-h)$ and $f_1(c-h)$ have opposite signs and $f(c+h)$ and $f_1(c+h)$ have like signs. Therefore the signs of $f(x)$ and $f_1(x)$ before we pass through c, are either $+\ -$ or $-\ +$ while their signs after we pass through c, are either $-\ -$ or $+\ +$; in any case, there is a loss of one change of sign.

We thus conclude that whenever we pass through a real root of $f(x)=0$ in the positive direction of the real variable x, there is a loss of one change of sign and that the loss occurs under no other circumstances. Therefore *the difference between the number of changes of sign in the series of signs of* $f(a)$, $f_1(a),\ldots,$ $f_n(a)$ *and the number of changes of sign in the series of signs of* $f(b)$, $f_1(b)\ldots$ $f_n(b)$, *gives the exact number of real roots of the equation that lie between* a *and* b.

Cor. 1. To obtain the total number of real roots of an equation $f(x)=0$, we substitute a numerically large negative number and a numerically large positive number, then the number of changes of sign lost, gives the total number of real roots.

Example 1. Calculate the Sturm's functions for the equation $x^5-5x+1=0$ and find the total number of real roots.

1st method: The Sturm's functions are easily found to be

$$x^5-5x+1\ ;\ x^4-1;\ 4x-1\ ;\ \text{and}\ 1-(\tfrac{1}{4})^4.\ *$$

Putting $-\infty$ for x in these functions, we get the series of signs

$$-\ +\ -\ +\ .$$

Putting ∞ for x in the same functions, we get the series of signs

$$+\ \ +\ \ +\ \ +\ .$$

Hence three changes of sign are lost and so there are three real roots of the equation.

2nd method: The same conclusion may be reached thus:-

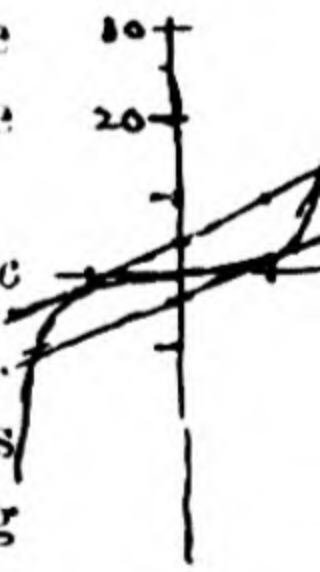
Denote x^5-5x+1 by $f(x)$; then $f'(x)=5(x^4-1)=0$ has two real roots -1, 1. These roots, by § 6·6, separate those of the given equation. Putting successively $-\infty$, -1, 1, ∞ for x in $f(x)=0$, we have the series of signs

$$-\ ,\ +\ ,\ -\ ,\ +\ .$$

Hence, by § 6·5, the equation has three real roots.

3rd method. The problem also lends itself to a simple graphical treatment.

The real roots of $f(x)=0$ are clearly the x-coordinates of the points of intersection of the curve $y=x^5$ and the straight line $y=5x-1$.



Now the line $y=5x+c$ touches the curve, $y=x^5$, if $x^5=5x+c$ has equal roots *i. e.* if $x^4=1$ and $x^5=5x+c$ have a common root. The condition for this, is that c should be equal to 4 and -4. It is easily seen from a figure, that any line parallel to $y=5x$ and lying between $y=5x\mp 4$ meets the curve in three points; while all other lines parallel to them, cut the curve in one pt. only. Hence $x^5-5x+1=0$ has three real roots.

* *N. B.* As we are concerned only with the sign of the leading coefficient of a Sturm's function, to avoid fractions, we may multiply or divide the remainders by suitable *positive* numbers.

Example 2. Discuss the nature of the roots of the equation $f(x) \equiv (n-1)x^n - npx^{n-1} + q = 0$, p and q being positive numbers.

Suppressing the factor $n(n-1)$, we easily find that

$$f_1(x) = x^{n-1} - px^{n-2}.$$

By division, we have

$$(n-1)x^n - npx^{n-1} + q = [\,(n-1)x - p\,](x^{n-1} - px^{n-2}) - p^2x^{n-2} + q;$$

Hence $$f_2(x) = p^2x^{n-2} - q.$$

Again, by actual division, we have

$$x^{n-1} - px^{n-2} = (p^{-2}x - p^{-1})(p^2x^{n-2} - q) + qp^{-2}x - qp^{-1}$$

Hence $$f_3(x) = -q^{-1}p^2(qp^{-2}x - qp^{-1}) = -x + p.$$

Lastly the remainder, when $f_2(x)$ or $p^2x^{n-2} - q$ is divided by $-x+p$ is, $p^n - q$; Hence $$f_4(x) = q - p^n.$$

Thus all the Sturm's functions are

$$(n-1)x^n - npx^{n-1} + q,\ x^{n-1} - px^{n-2},\ p^2x^{n-2} - q,\ -x + p,\ q - p^n.$$

Case I. If n is odd and $q - p^n$ is positive, the signs, when $-\infty$ and ∞ are substituted in the Sturm's functions are

$$- \quad + \quad - \quad + \quad +$$

and

$$+ \quad + \quad + \quad - \quad +.$$

Thus only one change of sign is lost; therefore there is only one real root in this case.

Case II. If n is odd and $q - p^n$ is negative, the series of signs are

$$- \quad + \quad - \quad + \quad -$$

and

$$+ \quad + \quad + \quad - \quad -.$$

In this case, three changes of sign are lost; hence there are three real roots.

Case III. If n is even and $q - p^n$ is negative, the series of signs are

$$+ \quad - \quad + \quad + \quad -$$

and

$$+ \quad + \quad + \quad - \quad -$$

Two changes of sign are lost, and therefore there are two real roots in this cuse.

Case IV. Lastly, if n is even, and $q - p^n$ is positive the series of sig ns are

$$+ \quad - \quad + \quad + \quad +$$

$$+ \quad + \quad + \quad - \quad +$$

In this case no change of sign is lost; hence there is no real root.

A much simpler solution may be obtained by applying the 2nd method of *example*, 1, above.

Examples. VI. f.

Examine the nature of the roots of the equations;

(1) $20x^3-30x^2+12x-1=0$. (2) $2x^3+3x^2+6x+1=0$.

(3) $2x^3+6x^2+6x+2=0$. (4) $4x^4+4x^3+1=0$.

(5) Prove that all the roots of $x^3-3px^2-3x+p=0$ are real for all real values of p.

(6) The roots of the equation $4x^3-3x+k=0$ are all real if

$$|k| \geqslant 1.$$

(7) The equation $x^4+4x+k=0$ has two real roots or none according as k is not, or is, greater than 3. Solve the problem graphically.

(8) Apply Sturm's theorem to the cubic $x^3+3bx+c=0$ and prove that it has one or three real roots according as c^2+4b^3 is or is not greater than zero. (*c.f* § 5·811).

(9) Prove that the equation $2x^5-5x^2+k=0$ has three real roots if $0<k\geqslant 3$. For other values of k the equation has only one real root.

6·9. Horner's method of approximating to a root.

Suppose we know by inspection or otherwise that a root a of $f(x)=0$ lies between two positive consecutive integers b and $b+1$. Reduce the roots of the equation by b (§5.64) and then multiply them by 10 (§5·63). Let the transformed equation be $f_1(x)=0$. Denote the root of $f_1(x)=0$ corresponding to the root a of $f(x)=0$ by a_1. Then it is clear that

$$a=b+\tfrac{1}{10}a_1 . \qquad (1)$$

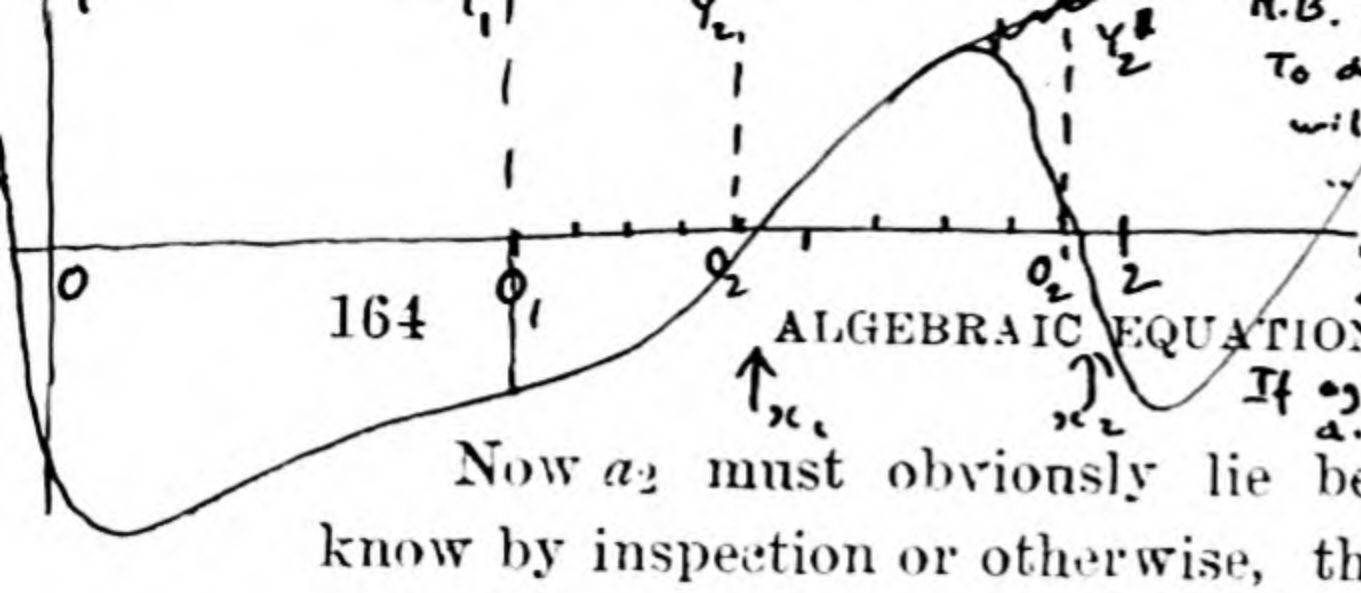
Now a_1 must obviously lie between 0 and 10. Suppose we know by inspection or otherwise, that a_1 of $f_1(x)=0$ lies between two consecutive integers b_1 and b_1+1. Then, from (1), we have

$$b+\tfrac{1}{10}b_1 < a < b+\tfrac{1}{10}b_1+\tfrac{1}{10}. \qquad (2)$$

Thus the extreme quantities of (2) differ from the root by less than $\frac{1}{10}$

Reduce the roots of $f_1(x)=0$ by b_1 (§ 5·64) and then multiply them by 10 (§ 5·63). Let the transformed equation be $f_2(x)=0$. Denote the root of $f_2(x)=0$, corresponding to the root a_1 of $f_1(x)=0$ by a_2. Then it is clear that

$$a_1=b_1+\tfrac{1}{10}a_2. \qquad (3)$$



Now a_2 must obviously lie between 0 and 10. Suppose we know by inspection or otherwise, that a_2 of $f_2(x)=0$, lies between two consecutive integers b_2 and b_2+1. Then from (3), we have

$$b_1+\tfrac{1}{10}b_2<a_1<b_1+\tfrac{1}{10}b_2+\tfrac{1}{10} \qquad (4)$$

$\therefore$ from (1) and (4) we have

$$b+\tfrac{1}{10}b_1+\tfrac{1}{100}b_2<a<b+\tfrac{1}{10}b_1+\tfrac{1}{100}b_2+\tfrac{1}{100} \qquad (5)$$

Thus the two extreme quantities of (5) differ from the root by less than $\frac{1}{100}$.

Repeating this process, it is possible to get as close an approximation to a as we please.

Cor. *If a root of $f(x)=0$ is negative, we evaluate the corresponding positive root of $f(-x)=0$.* (§ 5·61).

Example 1. Calculate Sturm's functions for the equation $x^3=x+2$ and prove that it has only one real root lying between 1 and 2. Determine the real root correct to two places of decimals.

[*B. U. Hons. I*]

The Sturm's functions are easily found to be

$$x^3-x-2,\ 3x^2-1,\ x+3, \text{ and } -26.$$

When $-\infty$ and ∞ are put for x in these functions the values have the following series of signs

$$- \quad + \quad - \quad -$$

and

$$+ \quad + \quad + \quad - \ .$$

Thus, only one change of sign is lost, hence by § 6·82, the equation has only one real root. By substitution, it is easy to verify that the root lies between 1 and 2.

By applying the process of § 5·64, the equation, whose roots are equal to the roots of the given equation, reduced by 1, is found to be

$$x^3+3x^2+2x-2=0. \qquad (1)$$

Now by § 5·63, the equation whose roots are equal to those of (1) multiplied by 10, is found to be

$$x^3+30x^2+200x-2000=0. \qquad (2)$$

By inspection, we find that the real root of (2) lies between 5 and 6. The root of the given equation must therefore lie between 1·5 and 1·6.

Reduce the roots of (2) by 5 and then multiply them by 10, the transformed equation is then found to be

$$x^3+450x^2+57500x-125000=0. \qquad (3)$$

The real root of (3) is found by inspection to lie between 2 and 3. Hence the real root of the given equation must be between 1·52 and 1·53.

Operating on (3) similarly, we have

$$x^3+4560x^2+5931200x-8192000=0 \qquad (4)$$

We find by inspection that the real root of (4) lies between 1 and 2. Hence the real root of the given equation lies between 1·521 and 1·522.

The following scheme shows how the problem is worked out in practice :—

1	0	−1	−2
	1	1	0
1	0	−2000	
1	2	1875	
2	200	−125000	
1	175	+16808	
30	375	−8192000	
5	200	−2256239	
35	57500	−473711	
5	904	416030	
40	58404	−57681	
5	908	53489	
450	5931200	−4192	
2	5935761	3366	
452	5940323	−626	
2	594176	535	
454			
2			
4560			
4561			
4562			
4563			
4564			
4565			

Examples. VI. g.

1. Calculate the cube root of 7 to 3 places of decimals.

2. Calculate to three places of decimals, the positive root of the equation. $x^3+2x^2-x-3=0$. [*B. U. 1923 Hons.* I]

3. Calculate the positive root of the equation

$$x^3+x^2+x-1=0$$

to three places of decimals. [*B. U. 1921 Hons.* I]

4. Find the negative root of the equation

$$x^4-12x^2+12x-3=0$$

correct to four places of decimals. [*B. U. 1926 Hons.* I]

5. $(1-\sqrt{\frac{26}{27}})^{\frac{1}{3}}+(1+\sqrt{\frac{26}{27}})^{\frac{1}{3}}$ is the positive real root of the cubic $x^3-x-2=0$. Hence calculate its value to three places of decimals by Horner's method.

6. Apply the method suggested in the last example to evaluate $\sqrt{-1+\sqrt{(2+\sqrt{5})}}$, and $\sqrt{1+(1+\sqrt{2})^{\frac{1}{3}}}$.

7. The equation $x^3-15x^2+60x-50=0$ has three roots lying between 0 and 10. Calculate each to two places of decimals.

References.

F. Cajori. *Theory of Equations*, Chaps. I (omit § 26), II, III (omit §§ 39–41, 50, 51,) IV, V, VII.

Burnside and Pantion. *Theory of Equations.* vol. I, chaps. I, II, III, IV (omit § § 36–38, 43, 44,) VI (contains many alternative methods for solving cubic and biquadratic equations.) VII, VIII, X (Read only §§ 96,99,) XI (Read only §§ 107, 108,) Read also notes, A and B.

Examples. VI. h.

1. Prove that all the roots of the eqnation in x are real

$$f(x) \equiv \frac{p^2}{a^2-x}+\frac{q^2}{b^2-x}+\frac{r^2}{c^2-x}=1$$

If x_1, x_2, x_3 are its roots, establish the identity

$$\frac{p^2}{a^2-x}+\frac{q^2}{b^2-x}+\frac{r^2}{c^2-x}=1-\frac{(x-x_1)(x-x_2)(x-x_3)}{(x-a^2)(x-b^2)(x-c^2)}.$$

Deduce, by multiplying by a^2-x, etc. and putting $x=a^2$, etc.

$$p^2=-\frac{(x_1-a^2)(x_2-a^2)(x_3-a^2)}{(a^2-b^2)(a^2-c^2)}$$

$$q^2=-\frac{(x_1-b^2)(x_2-b^2)(x_3-b^2)}{(b^2-a^2)(b^2-c^2)},$$

$$r^2=-\frac{(x_1-c^2)(x_2-c^2)(x_3-c^2)}{(c^2-a^2)(c^2-b^2)},$$

and, by $x=0$

$$\frac{p^2}{a^2}+\frac{q^2}{b^2}+\frac{r^2}{c^2}=1-\frac{x_1x_2x_3}{a^2b^2c^2}.$$

2. Prove that $(2x+1)^{2n}-1-4nx-4nx^2$ is divisible by $x^2(x+1)^2$.

3. Prove that $(2x^2+x)^n-1+n-nx-2nx^2$ is divisible by
$$(x+1)^2(2x-1)^2.$$

4. Prove that the expression $f(x)\equiv 1+x^{2n}+(\quad)^{2n}$ is not divisible by $1+x+x^2$, divisible by $1+x+x^2$, divisible by $(1+x+x^2)^2$ according as n is of the form $3p$, $3p+1$, $3p+2$ respectively.
[*B. U. 1915 Problems*]

5. Find the nature of the roots of
$$x^5+2x^4+x^3-x^2-2x-1=0$$ [*B. U. 1925 Hons* I.]

6. Prove that if the equations
$$x^3+qx-r=0 \text{ and } rx^3-2q^2x^2-5qrx-2q^3-r^2=0$$
have a common root, the first equation will have a pair of equal roots and if each of these be a, find all the roots of the second equation. [*B. U. 1918 Hons* I.]

[We have the identity
$$6(rx^3-2q^2x^2-5qrx-2q^3-r^2)\equiv$$
$$[3(r+5)x-4q^2](3x^2+q)-3(15+r)(x^3+qx-r).$$
From this, the first part follows immediately. If a is a repeated root of $x^3+qx-r=0$, then $-2a$ must be the remaining root. Hence $q=-3a^2$, $r=-2a^3$. Putting these values in the second equation, it reduces to $(x-a)(x+5a)^2=0$.]

7. Show that the necessary and sufficient condition that the equations $ax^3+3bx+c=0$ and $a'x^3+3b'x+c'=0$ should have one common root is
$$27(bc'-b'c)(ab'-a'b)^2=(ca'-c'a)^3,$$
and that the common root is a double root of the cubic
$$2(ab'-a'b)x^3+(ac'-a'c)x^2+bc'-b'c=0.$$ [*B. U. 1923 Problems.*]

8. Prove that

$$(a)\quad \begin{vmatrix} x^4-1 & x^2-1 & x-1 \\ 4 & 2 & 1 \\ 4.3 & 2.1 & 0 \end{vmatrix} = -2(x-1)^3[\,(x-1)+4\,].$$

$$(b)\quad \begin{vmatrix} x^5-1 & x^2-1 & x-1 \\ 5 & 2 & 1 \\ 5.4 & 2.1 & 0 \end{vmatrix} = -2(x-1)^3[\,(x-1)^2+5(x-1)+10\,].$$

$$(c)\quad \begin{vmatrix} x^n-1 & x^2-1 & x-1 \\ n & 2 & 1 \\ n(n-1) & 2.1 & 0 \end{vmatrix}$$

$$= -2(x-1)^3[\,(x-1)^{n-3}+{}_nC_1(x-1)^{n-4}+{}_nC_2(x-1)^{n-5}\ldots+{}_nC_{n-3}\,].$$

cf. S—: ch 5-81 b

9. If $a_1>a_2>a_3>a_4\ldots>a_n$, then the equation

$$(x-a_1)(x-a_3)(x-a_5)\ldots+k(x-a_2)(x-a_4)(x-a_6)\ldots=0$$

has all its roots real, k being any real number.

10. If all the roots of $f(x)=0$ are real, those of $f(x)=m\,f'(x)$ are also all real.

11. When n is odd, the equation

$$\frac{x^n}{n}+\frac{x^{n-1}}{n-1}+\frac{x^{n-2}}{n-2}+\ldots+\frac{x^2}{2}+\frac{x}{1}+k=0.$$

has only one real root, and when n is even it has two real roots or none according as

$$\frac{1}{1.2}+\frac{1}{3.4}+\frac{1}{5.6}+\ldots+\frac{1}{n(n-1)} \gtrless k.$$

12. Prove that the equation

$$1+2x+3x^2+4x^3+\ldots+(n+1)x^n=0$$

has no real root or one according as n is even or odd. Further if S_r denotes the sum of the rth powers of its roots, show that

$$S_r=[\,(n+2)\div(n+1)\,]^r-2. \qquad (r<n+2).$$

13. The number of roots of an equation $f(x)=0$ algebraically greater than h is either equal to, or less by an even number, than the number of changes of sign in the series of signs of

$$f(h),\ f'(h),\ f''(h),\ldots f^{(n)}(h).$$

14. If $f(x)=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots+\frac{x^{2n}}{2n!}$

prove that the polynomial $f(x)\times f(-x)$ is even and that the coefficients are all positive. Hence shew that $f(x)=0$ has no real root.

CHAPTER VII.

TRIGONOMETRICAL APPLICATIONS OF THE THEORY OF EQUATIONS.

7·1. The following examples illustrate a method of dealing with trigonometrical equations that we often meet with in Analytical Geometry.

Example. 1. Prove that $l\cos x + m\sin x = n$ has two roots not differing from each other by a multiple of 2π If they are denoted by a and b, shew that

$$l = n\cos\tfrac{1}{2}(a+b)\sec\tfrac{1}{2}(a-b),\quad m = n\sin\tfrac{1}{2}(a+b)\sec\tfrac{1}{2}(a-b).$$

Making the substitution $z = \cos x + i\sin x$ and therefore the substitutions $2\cos x = z + z^{-1}$, $2i\sin x = z - z^{-1}$, the given equation reduces to

$$li(z+z^{-1}) + m(z-z^{-1}) = 2in$$

or

$$(li+m)z^2 - 2niz + (li-m) = 0. \qquad (1)$$

This equation is a quadratic and has therefore only two roots. The original equation must also have only two roots. If these are a and b, the corresponding roots of (1) must be

$$\cos a + i\sin a,\ \cos b + i\sin b.$$

Now the sum of the roots of (1), by §5.22, is $2ni/(li+m)$.

Hence
$$2ni = (li+m)[\cos a + i\sin a + \cos b + i\sin b]$$
$$= 2(li+m)\cos\tfrac{1}{2}(a-b)[\cos\tfrac{1}{2}(a+b) + i\sin\tfrac{1}{2}(a+b)]$$

$$\therefore\quad li+m = ni\sec\tfrac{1}{2}(a-b)[\cos\tfrac{1}{2}(a+b) - i\sin\tfrac{1}{2}(a+b)] \qquad (2)$$

Equating the real and imaginary parts of (2), the required results are obtained at once.

Example 2. Prove that the equation in x

$$a^2\cos^2 x + b^2\sin^2 x + 2ag\cos x + 2bf\sin x + c = 0$$

has four roots. If these roots are p, q, r, s, shew that

(i) $p+q+r+s =$ a multiple of 2π,

(ii) $-4ag = (a^2-b^2)[\cos p + \cos q + \cos r + \cos(p+q+r)]$,

(iii) $4bf = (a^2-b^2)[\sin p + \sin q + \sin r - \sin(p+q+r)]$,

and (iv) $2c = -(a^2+b^2) + (a^2-b^2)[\cos(q+r) + \cos(r+p) + \cos(p+q)]$.

Transforming the equation, as in the last example, by the substitution $z=\cos x+i\sin x$, we have

$$(a^2-b^2)z^4-4(bfi-ag)z^3+(2a^2+2b^2+4c)z^2$$
$$+4(ag+bfi)z+a^2-b^2=0 \quad (1)$$

This equation is of the 4th degree and has therefore four roots. Hence the original equation must also have four roots. If these are p, q, r, s, the roots of (1) are

$$\cos p+i\sin p,\ \cos q+i\sin q,\ \cos r+i\sin r, \text{ and } \cos s+i\sin s.$$

Now the product of the roots of (1) is, by § 5·22, equal to 1.

$$\therefore\ (\cos p+i\sin p)(\cos q+i\sin q)(\cos r+i\sin r)(\cos s+i\sin s)=1$$

i. e $\cos(p+q+r+s)+i\sin(p+q+r+s)=1=\cos 2r\pi+i\sin 2r\pi$:

where r is an integer;

$$\therefore \qquad p+q+r+s=2r\pi. \qquad (2)$$

Again the sum of the roots of (1) is, $4(bfi-ag)/(a^2-b^2)$;

$$\therefore\ 4(bfi-ag)=(a^2-b^2)[(\cos p+\cos q+\cos r+\cos s)$$
$$+i(\sin p+\sin q+\sin r+\sin s)]$$

Equating the real and imaginary parts and putting s for $2r\pi-p-q-r$ from (2), we have

$$-4ag=(a^2-b^2)[\cos p+\cos q+\cos r+\cos(p+q+r)\ ], \qquad (3)$$

and
$$4bf=(a^2-b^2)[\sin p+\sin q+\sin r-\sin(p+q+r)]. \qquad (4)$$

Further, the sum of the products of the roots of the equation (1) taken two together is, $(2a^2+2b^2+4c)/(a^2-b^2)$. Hence

$$2a^2+2b^2+4c=(a^2-b^2)[\Sigma\cos(q+r)+i\Sigma\sin(q+r)\ ];$$
$$\therefore\ 2a^2+2b^2+4c=(a^2-b^2)\Sigma\cos(q+r).$$

Now substituting $s=2r\pi-p-q-r$, we have

$$\Sigma\cos(p+q)=2[\cos(q+r)+\cos(r+p)+\cos(p+q)].$$

The above example contains a complete solution to the problem of four concyclic points on the ellipse $b^2x^2+a^2y^2=a^2b^2$. (*c. f.* Smith's *Conic Section*[169] § 136. Askwith's *Analytical Geometry*, § 213, and Salmon's *Conic Sections*,[220] § 231, *example* 8).

Example 3. Prove that the equation in x

$$ah\sec x-bk\operatorname{cosec} x=a^2-b^2$$

has four roots. If these are denoted by p, q, r, s prove that

(i) $p+q+r+s=$ an odd multiple of π,

(ii) $\sin(p+r)+\sin(r+q)+\sin(p+q)=0$

(iii) $2ah=(a^2-b^2)\,[\cos p+\cos q+\cos r-\cos(p+q+r)\,]$

and (iv) $-2bk=(a^2-b^2)\,[\sin p+\sin q+\sin r+\sin(p+q+r)]$.

After making the substitution $z=\cos x+i\sin x$, the equation reduces to

$$(a^2-b^2)z^4-(2ah-2bki)z^3+(2ah+2bki)z-(a^2-b^2)=0. \qquad (1)$$

It is a biquadratic and hence it has four roots. Therefore the the original equation must also have four roots. If they are p,q,r,s the corresponding roots of (1) must be

$\cos p+i\sin p$, $\cos q+i\sin q$, $\cos r+i\sin r$ and $\cos s+i\sin s$.

Now the product of the roots of (1) is, -1; hence,

$$(\cos p+i\sin p)(\cos q+i\sin q)(\cos r+i\sin r)(\cos s+i\sin s)=-1$$

$$\cos(p+q+r+s)+i\sin(p+q+r+s)=-1=\cos(2r-1)\pi+i\sin(2r-1)\pi$$

$$\therefore \quad p+q+r+s=(2r-1)\pi, \qquad (2)$$

where r is an integer.

Again the sum of the products of the roots of the equation (1), taken two together is zero; hence.

$$\Sigma(\cos p+i\sin p)(\cos q+i\sin q)=\Sigma\cos(p+q)+i\Sigma\sin(p+q)=0$$

$$\therefore \sin(p+q)+\sin(p+r)+\sin(p+s)$$

$$+\sin(r+s)+\sin(q+r)+\sin(q+s)=0. \qquad (3)$$

Now $s=(2r-1)\pi-p-q-r$; hence the condition (3), reduces to

$$\sin(q+r)+\sin(r+p)+\sin(p+q)=0.$$

Further the sum of the roots of the equation (1) is

$$(2ah-2bki)/(a^2-b^2).$$

$$\therefore \quad 2ah-2bki=(a^2-b^2)\,[\Sigma\cos p+i\Sigma\sin p]. \qquad (4)$$

Equating the real and imaginary parts on both sides of (4) and substituting for s from (2), we immediately get the remaining two results.

The above *Example* contains a complete (logically) solution to the problem of conormal points on the ellipse $b^2x^2+a^2y^2=a^2b^2$. (*cf.* Smith's *Conic Sections* § 139. Askwith's *Analytical Geometry of Conic Sections* §§ 166–170.)

Normals concr. to b,c,a concurrent? Yes, Salmon 220[10].
" " " b,d,a " *Examples.* VII. a.
" " " $\{b,c,d\}$ " $\{a,c,d\}$
cf.

1. If $\sin(b+c)+\sin(c+a)+\sin(a+b)=0$

and $\sin(b+d)+\sin(d+a)+\sin(a+b)=0$

prove that $a+b+c+d$ is an odd multiple of π and that

$$\sin(c+d)+\sin(d+b)+\sin(b+c)=0$$
$$\sin(a+c)+\sin(a+d)+\sin(c+d)=0 \ .$$

2. If the roots of $(1+\cos l)\tan x=\sin(l-x)+\cos x$ are a, b, c, $\pi+a$, $\pi+b$, $\pi+c$ prove that $a+b+c=\pi+l$.

3. If a, b, c, d satisfy the equation

$$(1+e\cos l)\sin x=(1+e\cos x)[e\sin l+\sin(l-x)]$$

prove that $a+b+c+d-2l=$ an odd multiple π.

7·2. Prove that the n quantities

$$\cos a,\ \cos(a+b),\ \cos(a+2b),\ \ldots\ldots\cos[a+(n-1)b]$$

where $nb=2\pi$, are the roots of the algebraic equation

$$f(x)\equiv 2^{n-1}x^n-\frac{n}{1!}2^{n-3}x^{n-2}+\frac{n(n-3)}{2!}2^{n-5}x^{n-4}$$
$$-\frac{n(n-4)(n-5)}{3!}2^{n-7}x^{n-6}+\ldots\ldots=\cos na,$$

By the formula (4) of § 4·4, $f(\cos\xi)\equiv\cos n\xi$;

$\therefore$ the given equation transforms into the trigonometrical equation

$$\cos n\xi=\cos na. \qquad (1)$$

The roots of (1) are obviously

$$\xi=a,\ a+b,\ a+2b,\ \ldots\ldots a+(n-1)b.$$

Hence the roots of $f(x)=\cos na$ are the n quantities

$$\cos a,\ \cos(a+b),\ \cos(a+2b),\ \ldots\ldots\cos[a+(n-1)b].$$

Cor. 1. Since the coefficient of the highest power of x in $f(x)$ is 2^{n-1}, we have the identity (§ 5·212)

$$f(x)-\cos na=2^{n-1}\prod_1^n[x-\cos(a+r-1b)]. \qquad (2)$$

Cor. 2. Putting $x=\cos x$ in the identity (2), and observing that $f(\cos x)=\cos nx$, we have the trigonometrical identity

$$\cos nx-\cos na=2^{n-1}\prod_1^n[\cos x-\cos(a+r-1b)]. \qquad (3)$$

Cor. 3. By means of the identity (2), it is possible to evaluate the symmetric functions of the n quantities

$$\cos a,\ \cos(a+b),\ \cos(a+2b)\ldots\ldots\cos[a+(n-1)b] \ .$$

(see § 5·5). Again by assigning particular values to x and a in (3), we can derive many interesting results.

7·21. Prove that

$$\cos 2nx = \left(1-\frac{\sin^2 x}{\sin^2 a}\right)\left(1-\frac{\sin^2 x}{\sin^2 3a}\right)\left(1-\frac{\sin^2 x}{\sin^2 5a}\right)\cdots\cdots\left(1-\frac{\sin^2 x}{\sin^2(2n-1)a}\right)$$

where $4na=\pi$.

This theorem can be deduced from Cor. 2 of § 7·2. It is however more convenient to establish it directly.

The equation $\cos 2nx=0$ is obviously satisfied by $2n$ values of x given by $x=\pm ra$ $[r=1, 3, 5,\ldots(2n-1)]$. Further $\cos 2nx$ can be expanded into a polynomial in $\sin^2 x$. Hence we have the identity of the form

$$\cos 2nx = k\left(1-\frac{\sin^2 x}{\sin^2 a}\right)\left(1-\frac{\sin^2 x}{\sin^2 3a}\right)\left(1-\frac{\sin^2 x}{\sin^2 5a}\right)\cdots\left(1-\frac{\sin^2 x}{\sin^2(2n-1)a}\right).$$

Now k is seen to be 1, by putting $x=0$ on both sides.

7·22. Prove that

$$\sin(2n+1)x = (2n+1)\sin x \prod_{r=1}^{r=n}\left\{1-\frac{\sin^2 x}{\sin^2 ra}\right\}$$

where $(2n+1)a=\pi$ and $r=1, 2, 3,\ldots n$.

It is easy to see that $\sin(2n+1)x/\sin x=0$ is satisfied by values $\mp a, \mp 2a, \mp 3a,\ldots \mp na$ of x. Further $\sin(2n+1)x/\sin x$ can be expanded into a polynomial in $\sin^2 x$ with $2n+1$ as the term independent of $\sin^2 x$ [see (14) of § 4·4]. Hence we have

$$\sin(2n+1)x/\sin x = (2n+1)\prod_{r=1}^{r=n}\left\{1-\frac{\sin^2 x}{\sin^2 ra}\right\}.$$

Example 1. Form the equation whose roots are $4\sin^2 a$, $4\sin^2 2a$, $4\sin^2 3a$ and shew that $2^3 \sin a.\ \sin 2a.\ \sin 3a=\sqrt{7}$, $7a$ being equal to π.

If x represents any one of the angles $\mp a, \mp 2a, \mp 3a$ then $\sin 7x/\sin x=0$. Now by (12) of § 4·4, we have the identity

$$(-1)^3\sin 7x/\sin x = (2\sin x)^6-7(2\sin x)^4+14(2\sin x)^2-7;$$

Hence $4\sin^2 a$, $4\sin^2 2a$, $4\sin^2 3a$, satisfy the equation

$$y^3-7y^2+14y-7=0. \qquad (1)$$

Now the product of the roots of the equation (1) is 7;

$$\therefore \quad 4^3\sin^2 a.\ \sin^2 2a.\ \sin^2 3a = 7. \qquad (2)$$

Taking square roots of both sides of (2) and observing that the sines of the acute angles a, $2a$, $3a$ are all positive, we have

$$2^3 \sin a.\ \sin 2a.\ \sin 3a = \sqrt{7}.$$

Example 2. Construct the equation whose roots are $2\cos a$, $2\cos 7a$, $2\cos 11a$, $2\cos 13a$, $15a$ being equal to π, and show that $\sin 6^0$ is a root of the equation $16x^4+8x^3-16x^2-8x+1=0$.

[*B. U. 1920 Hons* II]

We observe that $5y$ where y represents any one of the angles a, $7a$, $11a$, $13a$ is of the form $2r\pi \pm \frac{1}{3}\pi$; hence, they satisfy the equation

$$2\cos 5y - 1 = 0. \qquad (1)$$

Now by (4) of § 4·4, we have the identity

$$2\cos 5y = (2\cos y)^5 - 5(2\cos y)^3 + 5(2\cos y). \qquad (2)$$

Hence a, $7a$, $11a$, $13a$ satisfy the equation

$$(2\cos y)^5 - 5(2\cos y)^3 + 5(2\cos y) - 1 = 0 \qquad (3)$$

Therefore $2\cos a$, $2\cos 7a$, $2\cos 11a$, $2\cos 13a$, are four of the root of

$$x^5 - 5x^3 + 5x - 1 = 0. \qquad (4)$$

Now it is easy to see that $y = \frac{1}{3}\pi$ satisfies (1) and therefore (3) The corresponding root of (4) is $2\cos\frac{1}{3}\pi$ or 1.

Dividing the left hand member of (4.) by the corresponding factor $x-1$ and equating the quotient to zero, we have the required equation

$$x^4 + x^3 - 4x^2 - 4x + 1 = 0 \qquad (5)$$

Now $\sin 6^0 = \cos 7a$; hence $\sin 6^0$ satisfies

$$(2x)^4 + (2x)^3 - 4(2x)^2 - 4(2x) + 1 = 0$$

or

$$16x^4 + 8x^3 - 16x^2 - 8x + 1 = 0.$$

Example 3. Construct an equation whose roots are the values of $\cos\frac{r\pi}{13}$ for $r = 1, 3, 5, \ldots 11$. Deduce or otherwise prove that

$$\cos^2 a + \cos^2 3a + \cos^2 5a + \ldots + \cos^2 11a = 11/4$$

and

$$\sec^2 a + \sec^2 3a + \sec^2 5a + \ldots\ldots + \sec^2 11a = 84$$

where $13a = \pi$. [*B. U. 1919 Hons. II*].

We observe that $13x$ where x represents any one of the angles a, $3a$, $5a$, $\ldots$ $11a$, is an odd multiple of π; hence all these angles satisfy the equation $\cos 7x + \cos 6x = 0$.

Now by (4) of § 4·4, we have

$$2\cos 7x = (2\cos x)^7 - 7(2\cos x)^5 + 14(2\cos x)^3 - 7(2\cos x)$$

and $$2\cos 6x = (2\cos x)^6 - 6(2\cos x)^4 + 9(2\cos x)^2 - 2.$$

Hence the equation (1) is transformed into

$$(2\cos x)^7 + (2\cos x)^6 - 7(2\cos x)^5 - 6(2\cos x)^4 + 14(2\cos x)^3 + 9(2\cos x)^2 - 7(2\cos x) - 2 = 0.$$

$\therefore$ $\cos a$, $\cos 3a$, $\cos 5a$, $\cos 7a$,...$\cos 11a$, are six of the roots of

$$(2y)^7 + (2y)^6 - 7(2y)^5 - 6(2y)^4 + 14(2y)^3 + 9(2y)^2 - 7(2y) - 2 = 0. \quad (2)$$

In addition to the six values given above, the value $x = \pi$ also satisfies (1). The corresponding root of (2) is $\cos\pi$ or -1.

The required equation is, therefore, obtained by dividing the left hand member of (2) by $y+1$ or $2y+2$ and then equating the quotient to zero. By the method of Synthetic Division, it is easily found to be

1	1	−7	−6	14	9	−7	−2
1	−1	−6	4	6	−3	−1	0

$$(2x)^6 - (2x)^5 - 5(2x)^4 + 4(2x)^3 + 6(2x)^2 - 3(2x) - 1 = 0. \quad (3)$$

Now the sums of products of the roots of (3) taken one at a time and two at a time are respectively $\frac{1}{2}$ and $-5/4$

$$\therefore \quad \Sigma\cos^2 a = (\Sigma\cos a)^2 - 2\Sigma\cos a\cos 3a = 11/4.$$

Again the sums of products of the reciprocals of the roots of the equation (3), taken one at a time and two at a time, are respectively, $-6, -24$.

$$\therefore \quad \Sigma\sec^2 a = (\Sigma\sec a)^2 - 2\Sigma\sec a\sec 3a = 36 + 48 = 84.$$

N. B. $a, 3a, \ldots 11a$, satisfy the equation in x, $\cos 13x + 1 = 0$. This equation however leads to an equation of the 13th degree in $\cos x$ and each of its roots, $\cos a$, $\cos 3a$,... $\cos 11a$ is repeated twice. The solution would not be then so very neat; the student should therefore carefully note the form of the initial equation (1), above.

Example 4. If n is an odd integer, prove that

$$2^{\frac{1}{2}(n-1)}.\sin 2a.\ \sin 4a \ldots \sin(n-1)a = \sqrt{n}.$$

Prove that

$$[\cos b - \cos 2(n+1)b][\cos 3b - \cos 2(n+1)b]\ldots[\cos(2n-1)b - \cos 2(n+1)b]$$
$$= \left(\frac{4n+1}{2n+1}\right)^{\frac{1}{2}} \sin c.\ \sin 2c.\ \sin 3c \ldots \sin nc.$$

where $2n\,a = (4n+1)\,b = (2n+1)\,c = \pi$

[*B. U. 1919 Hons. II.*]

We observe that n times any one of the angles $\pm 2a, \pm 4a, \ldots\ldots \ldots \pm (n-1)a$ is a multiple of π; hence all these angles satisfy the equation

$$\sin nx / \sin x = 0. \qquad (1)$$

Now by (14) of § 4·4 we have the identity

$$\sin nx/\sin x = \frac{n}{1!} - \frac{n(n^2-1^2)}{3!}\sin^2 x + \frac{n(n^2-1^2)(n^2-3^2)}{5!}\sin^4 x - \ldots\ldots + (-1)^{\frac{1}{2}\overline{n-1}} 2^{n-1}\sin^{n-1} x.$$

Hence $\sin^2 2a$, $\sin^2 4a, \ldots \sin^2 (n-1)a$ satisfy the equation

$$\frac{n}{1!} - \frac{n(n^2-1^2)}{3!}x + \frac{n(n^2-1)(n^2-3^2)}{5!}x^2 + \ldots + (-1)^{\frac{1}{2}\overline{n-1}} 2^{n-1} x^{\frac{1}{2}\overline{n-1}} = 0. \qquad (2)$$

The product of the roots of (2) is $n\,2^{1-n}$;

$$\therefore 2^{n-1} \sin^2 2a . \sin^2 4a \ldots \sin^2 (n-1)a = n. \qquad (3)$$

Taking square roots of both sides and observing that $\sin 2a$, $\sin 4a, \ldots \sin (n-1)a$ are all positive, we have the required result.

To prove the second part, we observe that $4n+1$ is an odd number; hence from (3), we have

$$\sqrt{(4n+1)} = 2^{2n} \sin b . \sin 2b \sin 3b \ldots \sin nb . \sin (n+1)b \ldots \sin 2nb. \qquad (4)$$

By regrouping the factors of the right hand member of (4), we have

$$\sqrt{(4n+1)} = 2^n [2\sin nb \sin (n+1)b][2\sin (n-1)b \sin (n+2)b] \ldots$$
$$= 2^n [\cos b - \cos (2n+1)b][\cos 3b - \cos 2(n+1)b] \ldots \qquad (5)$$

Now since $2n+1$ is odd, we have similarly

$$\sqrt{(2n+1)} = 2^n \sin c \sin 2c \sin 3c \ldots \sin nc. \qquad (6)$$

From (5) and (6) the required result is easily obtained.

Examples. VII. b.

Prove the following identities:-

1. $x^2 - x - 1 = (x - 2\cos \frac{1}{5}\pi)(x - 2\cos \frac{3}{5}\pi)$.

2. $y^3 - y^2 - 2y + 1 = (y - 2\cos \frac{1}{7}\pi)(y - 2\cos \frac{3}{7}\pi)(y - 2\cos \frac{5}{7}\pi)$.

3 (i) $\sec a + \sec (a + \frac{2}{3}\pi) + \sec (a + \frac{4}{3}\pi) = -3 \sec 3a$.

(ii) $\sec a \sec (a + \frac{2}{3}\pi) \sec (a + \frac{4}{3}\pi) = 4 \sec 3a$.

4 Prove that $\cos 20^0$, $\cos 100^0$ $\cos 140^0$, are the roots of

$$8x^3 - 6x - 1 = 0.$$

5 Form the equation whose roots are $\cos a, \cos 7a, \cos 9a, \cos 17a$ where $20a = \pi$.

[Observe that $5ra$ is of the form $2n\pi \mp \frac{1}{4}\pi$ $(r = 1, 7, 9, 17)$]

6. Prove that

$$\Sigma \sec^2 ra = 192, \quad \Sigma (1+2\cos ra)^{-1} = -3$$

where $21a = \pi$, r being an odd number less than and prime to 21.

7. Establish the identity

$$x^4 - 8x^3 + 19x^2 - 12x + 1$$
$$= (x - 4\cos^2 \tfrac{1}{20}\pi)(x - 4\sin^2 \tfrac{1}{20}\pi)(x - 4\cos^2 \tfrac{3}{20}\pi)(x - 4\sin^2 \tfrac{3}{20}\pi).$$

If $f(x)$ denotes the left hand polynomial, shew that

$$f(x) = f(4-x).$$

8. If n is odd and $2na = \pi$. prove that

(i) $\cos^2 2a + \cos^2 4a + \cos^2 6a + \ldots\ldots + \cos^2 (n-1)a = (n-2)/4$,

(ii) $\sec^2 2a + \sec^2 4a + \sec^2 6a + \ldots\ldots + \sec^2 (n-1)a = (n^2-1)/2$,

and (iii) $\sec 2a \sec 4a \sec 6a \ldots\ldots \sec (n-1)a = 2^{\frac{1}{2}n-1}$.

9. In the same notation prove that

$$1 = 2^{\frac{1}{2}n-1} \sin a \sin 3a . \sin 5a \ldots\ldots \sin (n-2)a.$$

[Write $x = 0$ and $na = \pi$ in the identity (3) of § 7·2,]

7·3. *If* $nb = \pi$, *form the equation whose roots are*

$$\tan a,\ \tan(a+b),\ \tan(a+2b), \ldots \tan[a+(n-1)b]$$

Since $n(a+rb) = na + r\pi$ where r is a positive integer, $a+rb$ satisfies the equation $\tan nx = \tan na$. (1)

Now by §4·23, we have the identity

$$\tan nx (1 - {}_nC_2 \tan^2 x + {}_nC_4 \tan^4 x - \ldots) = {}_nC_1 \tan x - {}_nC_3 \tan^3 x + {}_nC_5 \tan^5 x - \ldots$$

Hence a, $a+b$, $a+2b, \ldots a+(n-1)b$ satisfy the equation

$$\tan na (1 - {}_nC_2 \tan^2 x + {}_nC_4 \tan^4 x - \ldots)$$
$$= {}_nC_1 \tan x - {}_nC_3 \tan^3 x + {}_nC_5 \tan^5 x - \ldots. \quad (2)$$

Therefore $\tan a$, $\tan(a+b)$, $\tan(a+2b)$, ...satisfy

$$\tan na\ (1 - {}_nC_2 y^2 + {}_nC_4 y^4 - \ldots) = {}_nC_1 y - {}_nC_3 y^3 + {}_nC_5 y^5 - \ldots \quad (3)$$

[The equation (3) may also be written thus

$$\sin na\ [(1+yi)^n + (1-yi)^n] = -i[(1+yi)^n - (1-yi)^n] \cos na. \quad (3')]$$

Since a, $a+b$, $a+2b, \ldots$ $a+(n-1)b$ are such that the difference between any two of them is less than π, their tangents are the distinct n roots of the equation (3).

Example 1. Form the equation whose roots are $\tan^2 a$, $\tan^2 2a$, $\tan^2 3a$, $\tan^2 4a$ where $9a = \pi$. [*B. U. 1924 Hons. II*]

$\mp a, \mp 2a, \mp 3a, \pm 4a$ are clearly the roots of $\tan 9x = 0$. ... (1)

Now by §4·23, we have identity

$$\tan 9x\,(1 - {}_9C_2 \tan^2 x + {}_9C_4 \tan^4 x - \ldots)$$
$$= {}_9C_1 \tan x - {}_9C_3 \tan^3 x + \ldots + {}_9C_9 \tan^9 x.$$

Hence $\mp \tan a, \mp \tan 2a, \mp \tan 3a, \mp \tan 4a$ are 8 of the roots of

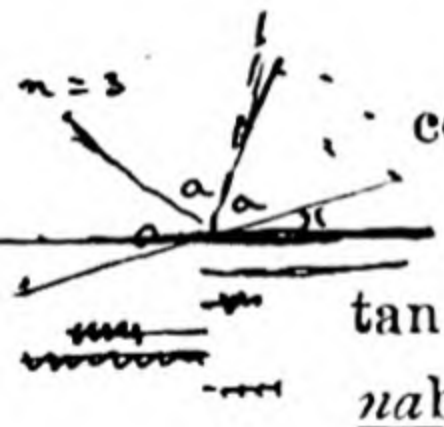
$$x\,(1 - {}_9C_1 x^2 + {}_9C_3 x^4 - {}_9C_5 x^6 + {}_9C_7 x^8 - {}_9C_9 x^{10}) = 0. \qquad (2)$$

Now the value, $x = 0$ satisfies (1); The corresponding root of (2) is zero; removing the factor x corresponding to this root from the left hand polynomial of (2), writing x for x^2 and then equating the result to zero, we have the required equation

$${}_9C_1 - {}_9C_3 x + {}_9C_5 x^2 - {}_9C_7 x^3 + {}_9C_9 x^4 = 0 \qquad (3)$$

Example 2. Show that $n \cot nx$ is equal to

$$\cot x + \cot(x+a) + \cot(x+2a) + \cot(x+3a) \ldots + \cot[x + (n-1)a]$$



What is the value of

$$\tan x + \tan(x+a) + \tan(x+2a) + \ldots + \tan[x + (n-1)a]$$

na being equal to π. [*B. U. 1919 Pass* I.]

As in § 7·3, $\tan x$, $\tan(x+a)$, $\tan(x+2a)$, ... are the roots of

$$\tan nx\,(1 - {}_nC_2 y^2 + {}_nC_4 y^4 - {}_nC_6 y^6 + \ldots) = {}_nC_1 y - {}_nC_3 y^3 + {}_nC_5 y^5 - \ldots (1)$$

Now the sum of the reciprocals of the roots of (1) is $n \cot nx$:

$$\therefore \cot x + \cot(x+a) + \cot(x+2a) + \ldots + \cot[x + (n-1)a] = n \cot nx.$$

If n is even, the equation (1) can be written thus

$$\tan nx\,(1 - {}_nC_2 y^2 + {}_nC_4 y^4 - \ldots + (-1)^{\frac{1}{2}n} y^n) = {}_nC_1 y - {}_nC_3 y^3 + {}_nC_5 y^5 - \ldots$$

Hence in this case

$$\tan x + \tan(x+a) + \tan(x+2a) + \ldots + \tan[x + (n-1)a] = -n \cot nx.$$

If n is odd the equation (1) can be written thus:-

$$\tan nx\,(1 - {}_nC_2 y^2 + {}_nC_4 y^4 - \ldots + (-1)^{\frac{1}{2}(n-1)}\, {}_nC_{n-1} y^{n-1})$$
$$= {}_nC_1 y - {}_nC_3 y^3 + \ldots + (-1)^{\frac{1}{2}(n-1)}\, {}_nC_n y^n.$$

Hence in this case

$$\tan x + \tan(x+a) + \tan(x+2a) + \ldots + \tan[x + (n-1)a] = n \tan nx.$$

Example 3. Show that when n is an odd positive integer greater than 2, the sum of the products taken two together of the $n-1$ quantities $\tan(r\pi/n)$, $(r=1, 2, 3, \ldots n-1)$ is $-\frac{1}{2}n(n-1)$.

[*B. U.* 1923 *Hons* II.]

All the angles $r\pi/n$, $r=1, 2, 3, \ldots n-1$ satisfy the equation

$$\tan nx \div \tan x = 0. \qquad (1)$$

Hence their tangents are roots of

$${}_nC_1 - {}_nC_3x^2 + {}_nC_5x^4 - \ldots + (-1)^{\frac{1}{2}(n-1)}\,{}_nC_n x^{n-1} = 0. \qquad (\S\ 7.3)$$

∴ the sum of the products of the roots taken two together is $-{}_nC_2$.

Example 4. Prove that

(i) $\operatorname{cosec}^2 a + \operatorname{cosec}^2 3a + \operatorname{cosec}^2 5a + \ldots$ to n terms $= 2n^2$,

(ii) $\operatorname{cosec}^4 a + \operatorname{cosec}^4 3a + \operatorname{cosec}^4 5a + \ldots$ to n terms $= \frac{4}{3}n^2(2n^2+1)$

where $na = \pi$. [*B. U. 1926 Problems*]

Now $2n(2r-1)a = (2r-1)\frac{1}{2}\pi$, $(r=1, 2, 3 \ldots n)$, Hence, all these angles satisfy the equation

$$\tan 2ny = \infty$$

or $$1 - {}_{2n}C_2t^2 + {}_{2n}C_4\,t^4 - {}_{2n}C_6\,t^6 + \ldots + (-1)^n\,{}_{2n}C_{2n}\,t^{2n} = 0$$

where $t = \tan y$. Putting $t^2 = x$, we get the equation

$$1 - {}_{2n}C_2x + {}_{2n}C_4x^2 - {}_{2n}C_6x^3 + \ldots + (-1)^n\,{}_{2n}C_{2n}x^n = 0 \qquad (1)$$

whose roots are $\tan^2 a$, $\tan^2 3a$, $\tan^2 5a, \ldots$ $\tan^2(2n-1)a$.

Now the sum of the reciprocals of the roots $= {}_{2n}C_2$,

∴ $\cot^2 a + \cot^2 3a + \cot^2 5a + \ldots$ to n terms $= {}_{2n}C_2$;

∴ $\operatorname{cosec}^2 a + \operatorname{cosec}^2 3a + \operatorname{cosec}^2 5a + \ldots$ to n terms

$$= 1 + \cot^2 a + 1 + \cot^2 3a + 1 + \cot^2 5a + \ldots$$

$$= n + {}_{2n}C_2 = 2n^2.$$

Again the sum of the products of the reciprocals of the roots of (1) taken two together is equal to ${}_{2n}C_4$.

∴ $\cot^4 a + \cot^4 3a + \cot^4 5a + \ldots = ({}_{2n}C_2)^2 - 2\,{}_{2n}C_4$.

∴ $\sum \operatorname{cosec}^4(2r-1)a = \sum [1 + \cot^2(2r-1)a]^2$

$$= n + 2\sum \cot^2(2r-1)a + \sum \cot^4(2r-1)a.$$

$$= n + 2n(2n-1) + \tfrac{1}{4}(2n)^2(2n-1)^2 - \tfrac{1}{3}n(2n-1)(n-1)(2n-3)$$

$$= \tfrac{4}{3}n^2(2n^2+1).$$

Examples VII. c.

Establish the following identities:

1. $5t^2-10t+1=5(t-\tan^2\frac{1}{10}\pi)(t-\tan^2\frac{3}{10}\pi)$.

2. $5y^2-20y+16=5(y-\sec^2\frac{1}{10}\pi)(y-\sec^2\frac{3}{10}\pi)$.

3. $t^3-21t^2+35t-7=(t-\tan^2\frac{1}{7}\pi)(t-\tan^2\frac{2}{7}\pi)(t-\tan^2\frac{3}{7}\pi)$.

4. $7t^3-56t^2+112t-64=7(t-\sec^2\frac{1}{14}\pi)(t-\sec^2\frac{3}{14}\pi)(t-\sec^2\frac{5}{14}\pi)$.

5. Show that the equation whose roots are $\tan ra, (r=1,4,7,13)$ where $15a=\pi$ is $t^4-6\sqrt{3}t^3+8t^2+2\sqrt{3}t-1=0$.

(Observe that $5x$ is of the from $r\pi+\frac{1}{3}\pi$; x representing any of the angles a, $4a$, $7a$, $13a$.)

6. Prove that
$(2\cos 6x+1)=\cos^6 x(\tan^2\frac{1}{9}\pi-\tan^2 x)(\tan^2\frac{2}{9}\pi-\tan^2 x)(\tan^2\frac{4}{9}\pi-\tan^2 x)$.

7. Prove that

$$\tan a\tan(a+b).\tan(a+2b).....\tan[a+(n-1)b]$$

is equal to $(-1)^{\frac{1}{2}n(n-1)}$ or $(-1)^{\frac{1}{2}n(n-1)}$ $\tan na$ according as n is even or odd, nb being equal to π.

8. Prove that

$$\cos nx=\cos^n x\prod_{0}^{\frac{1}{2}n-1}\left\{1-\frac{\tan^2 x}{\tan^2\frac{2r+1}{2n}}\right\}$$

where n is even. [*B. U. 1925 Hons* I.]

[Since n is even, $\cos nx\sec^n x$ can be expanded into a polynomial in $\tan^2 x$ (§ 4·23). Further the values of x given by $\cos nx=0$ are $\pm a$, $\pm 3a$, $\pm 5a$...where $2na=\pi$]

9. Prove that

$(\text{vers}a)^{-1}+(\text{vers}3a)^{-1}+(\text{vers}5a)^{-1}+$,.....+to n terms $=n^2$

$2na$ being equal to π. [*B. U. 1918 Hons* II.]

7·4. We shall now study some of the properties of the n th roots of unity.

Prove that 1, $\cos a+i\sin a$, $\cos 2a+i\sin 2a$, $\cos 3a+i\sin 3a$,..... $\cos(n-1)a+i\sin(n-1)a$ *are the* n *distinct* nth roots *of unity,* na *being equal to* 2π.

An nth root of unity is clearly a root of the algebraic equation

$$x^n=1. \qquad (1)$$

Now $1=\cos 2r\pi+i\sin 2r\pi=(\cos ra+i\sin ra)^n$ (§4·2)

and hence the equation (1) may be written thus

$$x^n=(\cos ra+i\sin ra)^n. \quad (2)$$

From this, we see that $\cos ra+i\sin ra$ is an nth root of unity. Putting $r=0, 1, 2,\ldots, n-1$, we have 1, $\cos a+i\sin a$, $\cos 2a+i\sin 2a$, $\ldots\cos(n-1)a+i\sin(n-1)a$ all of which are nth roots of unity (3)

Further no two of the n quantities (3) are equal; for if p, q are integers less than n, $(p-q)a$ is less than 2π and hence $\cos pa$, $\sin pa$, cannot be simultaneously equal to $\cos qa$, $\sin qa$. The n quantities (3) are therefore *the n nth roots of unity.*

7·41. *If* $\cos a+i\sin a$ *is denoted by* w, *then the n nth roots of unity may be written thus* $w^0, w^1, w^2, w^3,\ldots w^{n-1}$.

By Demoivre's Theorem $w^r=(\cos a+i\sin a)^r=\cos ra+i\sin ra$.

and so the two series

$$1,\ \cos a+i\sin a,\ \cos 2a+i\sin 2a,\ldots\cos(n-1)a+i\sin(n-1)a$$

$$w^0, w^1, w^2\ldots, w^{n-1}$$

are identical.

This result is only a particular case of the theorem:– If w is any special nth root of unity, then $w^0, w^1, w^2,\ldots w^{n-1}$ are the distinct nth roots of unity. An nth root of unity is called *a special nth root* if it is not an mth root of unity where the integer m is less than n. Such roots possess interesting properties. (Read '*The Theory of Equations*' by Burnside and Panton vol. I, §§ 46–53.)

7·411. *The complex nth roots of unity may be grouped into conjugate pairs,* ω^r, ω^{n-r}, $\ldots\ \omega^r, \omega^{-r}$ (whose sum is $2\cos r\frac{2\pi}{n}$)

$$w^{n-1}=w^n w^{-1}=w^{-1}=(\cos a+i\sin a)^{-1}=\cos a-i\sin a;$$

therefore w and w^{-1} are a pair of conjugate roots.

Similarly $w^{n-2}=w^n w^{-2}=w^{-2}=(\cos a+i\sin a)^{-2}=\cos 2a-i\sin 2a$.

therefore w^2, w^{-2} form another pair of conjugate roots.

In this way we get pairs of conjugate complex roots

$$(w, w^{-1});\ (w^2, w^{-2}),\ (w^3, w^{-3})\ldots\ldots\ldots$$

If n is even, one of the roots is $\cos\frac{1}{2}na+i\sin\frac{1}{2}na=-1$.

In this case 1 and –1 are the only real nth roots; the remaining roots can be grouped into $\frac{1}{2}(n-2)$ pairs

$$w,\ w^{-1};\ w^2,\ w^{-2};\ w^3,\ w^{-3};\ldots w^{\frac{1}{2}\overline{n-1}},\ w^{-\frac{1}{2}\overline{n-1}}.$$

If n is odd, 1 is the only real nth root, the remaining roots can be grouped into $\frac{1}{2}(n-1)$ conjugate pairs

$$w,\ w^{-1};\ w^2,\ w^{-2};\ w^3,\ w^{-3};\ldots w^{\frac{1}{2}\overline{n-1}}\ w^{-\frac{1}{2}\overline{n-1}}.$$

See also § 6·1.

7·412. *Prove that the sum of the mth powers of the n nth roots of unity is n or 0 according as m is or is not a multiple of n.*

If $m=pn$ where p is an integer and w^r is one of the nth roots, then

$$(w^r)^m = w^{rm} = w^{rpn} = (w^n)^{pr} = 1\ ;$$

hence the sum of the mth powers in this case is n.

If m is not a multiple of n, then $w^m \neq 1$;

$\therefore$ The sum of the mth powers

$$1+w^m+w^{2m}+w^{3m}+\ldots\text{to } n \text{ terms} = (1-w^{mn})/(1-w^m) = 0.$$

These results may also be easily deduced from §5·41.

7·42. *To find the n nth roots of any number c.*

All such numbers are clearly the roots of the equation

$$x^n = c. \tag{1}$$

Now by § 3·4, any number c can be put into the form

$$r\,(\cos p + i\sin p)$$

where r is a positive number and p, a real angle.

Further $\cos p + i\sin p = \cos(p+2m\pi) + i\sin(p+2m\pi)$ and

$$[\cos(b+ma)+i\sin(b+ma)]^n = \cos(p+2m\pi)+i\sin(p+2m\pi)$$

where $p=nb$, $na=2\pi$, and m an integer, $\therefore$ the equation (1) may be written thus

$$x^n = [r^{1/n}\cos(b+ma) + ir^{1/n}\cos(b+ma)]^n \tag{2}$$

wher $r^{1/n}$ denotes the arithmetical nth root of r. From (2), we see that

$$r^{1/n}[\cos(b+mq)+i\ \sin\ (b+ma)] \tag{3}$$

is an nth root of c for all values of m.

By putting $m=0,\ 1,\ 2,\ 3,\ldots n-1$, in (3), we have the n nth roots of c. These can be shown to be distinct in the manner of §7·4.

It should be noted that if we denote $r^{1/n}(\cos b+i\sin b)$ by y and $\cos a+i\sin a$ by w, then the nth roots of c are given by

$$yw^0,\ yw^1,\ yw^2,\ yw^3,\ldots yw^{n-1}.$$

7·421. As a particular case of § 7·42, let us consider *the* Laws II 141 29²
nth roots of −1, *or the roots of the equation* $x^n+1=0$. $x^n = \cos(2r-1)\pi$, etc.

Now $-1=\cos\pi+i\sin\pi$; ∴ $\cos a+i\sin a$ is one of the nth roots of −1 where $na=\pi$. Therefore by § 7·42, all the nth root of −1 are $\cos\frac{2r-1}{n}\pi$

$\cos a+i\sin a, \cos 3a+i\sin 3a, \cos 5a+i\sin 5a, \ldots \cos(2n-1)a+i\sin(2n-1)a$,

Like the nth roots of unity, we can group these roots into conjugate pairs. As with Thus take along with $\cos\frac{\pi}{n}$ $\cos\frac{2r-1}{n}\pi$

$\cos(2n-1)a+i\sin(2n-1)a=(\cos 2na+i\sin 2na)(\cos a-i\sin a)$

$=\cos a-i\sin a$;

∴ $\cos a\pm i\sin a$ is one pair of conjuate complex roots.

Similarly $\cos(2n-3)a+i\sin(2n-3)a=\cos 3a-i\sin 3a$;

∴ $\cos 3a\pm i\sin 3a$ is another pair of conjugate roots, and so on.

Now if n is even, $x^n+1=0$ has no real roots and thus in this case we have $\frac{1}{2}n$ pairs of complex conjugate roots

$\cos a\pm i\sin a, \cos 3a\pm i\sin 3a, \ldots\ldots \cos(n-1)a\pm i\sin(n-1)a$.

If n is odd, $x^n+1=0$ has only one real root,

$\cos na+i\sin na=\cos\pi+i\sin\pi=-1$.

The remaining $n-1$ roots can be grouped into $\frac{1}{2}(n-1)$ pairs

$\cos a\pm i\sin a, \cos 3a\pm i\sin 3a, \ldots\ldots \cos(n-2)a\pm i\sin(n-2)a$.

Example 1. Find the cube roots of $4+4\sqrt{(-3)}$. 110°

Suppose $4=r\cos p$ and $4\sqrt{3}=r\sin p$, then we easily have $r=8$ and $p=\tan^{-1}\sqrt{3}=\frac{1}{3}\pi$; hence if $\frac{1}{3}\pi=a$

$$4+4\sqrt{-3}=8(\cos 3a+i\sin 3a).$$

One of the cube roots is clearly $2(\cos a+i\sin a)$; the other two are $2(\cos a+i\sin a)(\cos 6a+i\sin 6a)=2(\cos 7a+i\sin 7a)$, $2\cos\frac{2\pi+\frac{\pi}{3}}{3}$

and $2(\cos a+i\sin a)(\cos 12a+i\sin 12a)=2(\cos 13a+i\sin 13a)$. $2\cos\frac{4\pi+\frac{\pi}{3}}{3}$

Camb II 29³ *Example* 2. Solve the equation $x^7-x^4+2x^3-2=0$.

[*B. U. 1924 Hons. II*]

$$x^7-x^4+2x^3-2=x^4(x^3-1)+2(x^3-1)=(x^3-1)(x^4+2).$$

∴ the required roots are given by $x^3=1$ and $x^4=-2$.

∴ they are the three cube roots of 1, 1, $\cos\frac{2}{3}\pi\mp i\sin\frac{2}{3}\pi$ and the four fourth roots of −2,

$$2^{\frac{1}{4}}\left(\cos\frac{2r-1}{4}\pi+i\sin\frac{2r-1}{4}\pi\right)$$

where $r=0, 1, 2, 3$.

Example 3. Find the equation whose roots are the twelve values of the expression

$$\frac{1\pm\sqrt{3}}{\sqrt{2}}\left(\cos ra+i\sin ra\right)$$

where r is an integer and $3a=\pi$. $a=\frac{\pi}{6}$ [*B U. 1919 Hons. II.*]

Denoting this expression by x, we have

$$x^6=\left(\frac{1\pm\sqrt{3}}{\sqrt{2}}\right)^6\left(\cos ra+i\sin ra\right)^6=\left(\frac{1\pm\sqrt{3}}{\sqrt{2}}\right)^6$$

The required equation is therefore

$$\left\{x^6-\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)^6\right\}\left\{x^6-\left(\frac{1-\sqrt{3}}{\sqrt{2}}\right)^6\right\}=0$$

or

$$x^{12}-52x^6+1=0.$$

Examples. VI. d.

Find all the values of the follwing:

1. $(-1)^{\frac{1}{2}}$. 2. $(-1)^{\frac{1}{4}}$. 3. $(i)^{\frac{1}{2}}$. 4. $(i)^{\frac{1}{3}}$

5. $(1+i)^{\frac{1}{2}}$ 6. $(1+i)^{\frac{1}{2}}+(1-i)^{\frac{1}{2}}$. 7. $4^{\frac{1}{4}}+2^{\frac{1}{2}}$.

8. $2^{\frac{1}{2}}.(-1)^{\frac{1}{2}}$

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7·5. *Resolve into factors* x^n-1.

Suppose n is even; then the roots of $x^n=1$ are the n nth roots of unity. In this case, they are in the notation of § 7·411

$$1,-1,\ w,\ w^{-1};\ w^2,\ w^{-2};\ldots w^{\frac{1}{2}n-1},\ w^{-\frac{1}{2}n+1},$$

The factors of x^n-1 corresponding to these pairs of roots are

$$(x-1)(x+1)=x^2-1$$

$$(x-w)(x-w^{-1})=x^2-(w+w^{-1})x+1=x^2-2x\cos a+1,$$

$$(x-w^2)(x-w^{-2})=x^2-(w^2+w^{-2})x+1=x^2-2x\cos 2a+1;$$

.....

$$(x-w^{\frac{1}{2}n-1})(x-w^{-\frac{1}{2}n+1})=x^2-(w^{\frac{1}{2}n-1}+w^{-\frac{1}{2}n+1})x+1$$
$$=x^2-2x\cos\tfrac{1}{2}(n-2)a+1$$

Now observing that the coefficient of x^n, in, x^n-1 is unity, we have the identity

$$x^n-1=(x^2-1)\prod^{\frac{1}{2}n-1}[x^2-2x\cos ra+1] \qquad (1)$$

When n is odd, the roots of $x^n-1=0$ are by 7·411,

$$1;\ w,\ w^{-1};\ w^2,\ w^{-2}:\ldots,w^{\frac{1}{2}\overline{n-1}},\ w^{\frac{1}{2}\overline{n-1}}.$$

Corresponding to these roots the factors of x^n-1 are $x-1$,

$$(x-w)(x-w^{-1}) = x^2-(w+w^{-1})x+1 = x^2-2x\cos a+1;$$

$$(x-w^2)(x-w^{-2}) = x^2-(w^2+w^{-2})x+1 = x^2-2x\cos 2a+1;$$

.....

$$(x-w^{\frac{1}{2}\overline{n-1}})(x-w^{-\frac{1}{2}\overline{n-1}}) = x^2-(w^{\frac{1}{2}\overline{n-1}}+w^{-\frac{1}{2}\overline{n-1}})x+1$$

$$= x^2-2x\cos\tfrac{1}{2}(n-1)a+1.$$

In this case we have the identity

$$x^n-1 = (x-1)\prod_{1}^{\frac{1}{2}(n-1)}[x^2-2x\cos(2\pi r/n)+1]. \qquad (2)$$

To find the factors of x^n-a^n where a is any number, we have only to write x/a for x in the above identities (1) and (2) and then to multiply both sides by a^n. Thus when n is even

$$x^n-a^n = (x^2-a^2)\prod_{1}^{\frac{1}{2}n-1}[x^2-2ax\cos(2r\pi/n)+a^2], \qquad (1')$$

and when n is odd

$$x^n-a^n = (x-a)\prod_{1}^{\frac{1}{2}(n-1)}[x^2-2ax\cos(2r\pi/n)+a^2]. \qquad (2')$$

7·51. Identities (1′) and (2′) of § 7·5 admit of the following geometrical interpretation.

$A_0, A_1, A_2, \ldots A_{n-1}$ are the vertices of a regular polygon inscribed in a circle, centre C and radius x. P is a pt. in the radius CA_0 distant a from C. Prove that

$$PA_0 \cdot PA_1 \cdot PA_2 \ldots PA_{n-1} = \pm(x^n-a^n)$$

positive or negative sign being chosen according as P is within or without, the circle.

Since $A_0, A_1, A_2, \ldots A_{n-1}$ are the vertices of a regular polygon of n sides, each of the arcs A_0A_1, A_1A_2, $\ldots A_{n-1}A_0$ subtends an angle, $2\pi/n$ at the centre. Hence from $\triangle PCA_1$, we have

$$PA_1^2 = x^2-2ax\cos(2\pi/n)+a^2.$$

Similarly from $\triangle s$ PCA_2, PCA_3, ...we have

$$PA_2^2 = x^2-2ax\cos(4\pi/n)+a^2$$

.....

and if n is even $PA_{\frac{1}{2}n-1}^2 = x^2-2ax\cos(n-2)\pi/n+a^2$

Again in this case $PA_{\frac{1}{2}n}=x+a$, $PA_0 = \pm(x-a)$

the upper or lower sign being taken according as P is within or without the circle.

Hence $PA_0 . PA_{\frac{1}{2}n} . PA_1^2 . PA_2^2 \ldots PA^2_{\frac{1}{2}n-1}$

$$= \pm (x^2-a^2) \prod_{1}^{\frac{1}{2}n-1} [x^2-2ax\cos(2r\pi/n)+a^2].$$

Now from the geometry of the figure it is evident that

$$PA_1 = PA_{n-1},\ PA_2 = PA_{n-2}, PA_3 = PA_{n-3}, \ldots$$

Thus the above result transforms into

$$PA_0 . PA_1 . PA_2 \ldots PA_{n-1} = \pm (x^2-a^2) \prod_{1}^{\frac{1}{2}n-1} [x^2-2ax\cos(2r\pi/n)+a^2]$$

$$= \pm (x^n-a^n).$$

If n is odd we can similarly prove that

$$PA_0 . PA_1^2 . PA_2^2 \ldots PA^2_{\frac{1}{2}(n-1)} = \pm (x-a) \prod_{1}^{\frac{1}{2}(n-1)} [x^2-2ax\cos(2r\pi/n)+a^2]$$

$$= \pm (x^n-a^n)$$

It is easy to see as before, from the geometry of the figure that $PA_1 = PA_{n-1}$, $PA_2 = PA_{n-2}, \ldots$ and so on. Hence in this case also the theorem holds.

7·6. *Resolve into factors* x^n+1.

If n is even, by §7·421, the roots of $x^n+1=0$ are given by

$$\cos a \pm i\sin a,\ \cos 3a \mp i\sin 3a, \ldots \cos(n-1)a \pm i\sin(n-1)a$$

where $na=\pi$. The factors of x^n+1 corresponding to these pairs of roots are

$$(x-\cos a-i\sin a)(x-\cos a+i\sin a) = x^2-2x\cos a+1;$$

$$(x-\cos 3a-i\sin 3a)(x-\cos 3a+i\sin 3a) = x^2-2x\cos 3a+1;$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$[x-\cos(n-1)a-i\sin(n-1)a][x-\cos(n-1)a+i\sin(n-1)a]$$

$$= x^2-2x\cos(n-1)a+1.$$

Now the leading coefficient of x^n+1 is 1; hence we have the identity

$$x^n+1 = (x^2-2x\cos a+1)(x^2-2x\cos 3a+1)\ldots[x^2-2x\cos(n-1)a+1].$$

Similarly when n is odd, by § 7·421, the roots of $x^n+1=0$ are the single real root -1, and the $\frac{1}{2}(n-1)$ pairs of roots

$$\cos a \pm i\sin a,\ \cos 3a \mp i\sin 3a, \ldots \cos(n-2)a \pm i\sin(n-2)a.$$

The corresponding factors of x^n+1 are, as before

$$x+1,\ x^2-2x\cos a+1,\ x^2-2x\cos 3a+1, \ldots x^2-2x\cos(n-2)a+1.$$

Hence when n is odd, we have

$$x^n+1=(x+1)(x^2-2x\cos a+1)(x^2-2x\cos 3a+1)\ldots \ldots[x^2-2x\cos(n-2)a+1.]$$

To factorise x^n+y^n, we write x/y for x in the above identities and then multiply both sides by y^n. We thus see that x^n+y^n is either

$$\prod_{1}^{n/2}[x^2-2xy\cos(2r-1)a+y^2] \text{ or } (x+y)\prod_{1}^{\frac{1}{2}(n-1)}[x^2-2xy\cos(2r-1)a+y^2]$$

according as n is even or odd.

7·61. The following is the geometrical interpretation of the latter result:

If $A_1, A_2, \ldots A_n$ are the vertices of a regular polygon inscribed in a circle and F, a pt. on the radius that bisects the chord $A_1 A_n$; then

$$PA_1 . PA_2 . PA_3 \ldots PA_{n-1} . PA_n = x^n + y^n,$$

where x is the radius and y the distance of P from the centre of the circle.

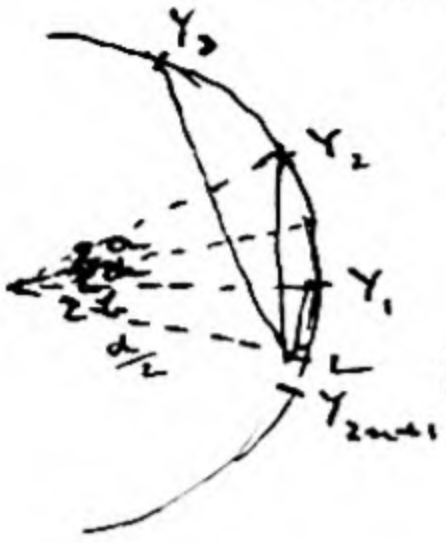
The proof is left as an exercise to the student. This and the result of § 7·51, are known as **Cote's properties of a** circle.

Example 1, A regular polygon $Y_1 Y_2 \ldots Y_{2n+1}$ is inscribed in a circle. Any pt. L on the arc $Y_{2n+1} Y_1$ is joined to the vertices $Y_1, Y_2, \ldots Y_{2n+1}$; prove that

$$(LY_1+LY_{2n+1})(LY_2+LY_{2n})\ldots(LY_n+LY_{n+2}) = (LY_{n+1})^n.$$

[*B. U. 1917 Hons.* II.]

Denote the angle subtended by the minor arc LY_1 at the circumference by b and let $2(2n+1)a=\pi$; then it is clear that LY_1, LY_2, LY_3...subtend angles b, $b+2a$, $b+4a$...and LY_{2n+1}, LY_{2n}, LY_{2n-1},...subtend angles $2a-b$, $4a-b$, $6a-b$,...at the circumference.



Hence if we denote the diameter of the circle by d we have

$$LY_1 = d\sin b,\ LY_{2n+1} = d\sin(2a-b),$$

$$\therefore \quad LY_1 + LY_{2n+1} = 2d\sin a\cos(a-b);$$

Similarly $LY_2 + LY_{2n} = 2d\sin 3a\cos(a-b);$

$$LY_3 + LY_{2n-1} = 2d\sin 5a\cos(a-b);$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Hence $(LY_1 + LY_{2n+1})(LY_2 + LY_{2n})\ldots(LY_n + LY_{n+1})$

$$= 2^n d^n \sin a.\ \sin 3a.\ \sin 5a\ldots\sin(2n-1)a\ \cos^n(a-b).$$

Now since $2n+1$ is an odd integer, we have by putting $x=1$ in (2) of §7·6, $1 = 2^n \sin a \sin 3a \sin 5a\ldots\sin(2n-1)a$.

Again it is clear from the geometry of the figure that LY_{n+1} subtends $2na+b$ at the circumference;

$$\therefore \quad LY_{n+1} = d\sin(2na+b) = d\cos(a-b).$$

The required result is now evident.

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7·7. *Resolve into factors* $x^{2n} - 2a^n x^n \cos np + a^{2n}$.

We observe that $x^{2n} - 2a^n x^n \cos np + a^{2n}$ is a quadratic expression in x^n and hence, putting it into the form

$$(x^n - a^n\cos np)^2 + a^{2n}\sin^2 np \text{ or } (x^n - a^n\cos np)^2 - i^2\sin^2 np.a^{2n}$$

we see that it has two factors

$$x^n - a^n(\cos np + i\sin np) \quad \text{and} \quad x^n - a^n(\cos np - i\sin np).$$

Since $\cos np + i\sin np = (\cos p + i\sin p)^n$, these factors may be put into the form $x^n - a^n b^n, x^n - a^n b^{-n}$ where $b = \cos p + i\sin p$.

Now if w is one of the nth roots of unity.

$$\cos(2r\pi/n) + i\sin(2r\pi/n)$$

it is clear that abw or $a[\cos(p + 2r\pi/n) + i\sin(p + 2r\pi/n)]$ is a root of $x^n - a^n b^n = 0$ and therefore $x - abw$ is a factor of $x^n - a^n b^n$. Similarly $x - ab^{-1}w^{-1}$ is a factor of $x^n - a^n b^{-n}$; hence $(x - abw)(x - ab^{-1}w^{-1})$ or

$$x^2 - ax(bw + b^{-1}w^{-1}) + a^2 = x^2 - 2ax\cos(p + 2r\pi/n) + a^2$$

is a factor of $x^{2n} - 2a^n x^n \cos np + a^{2n}$.

Taking into account all other nth roots of unity and observing that the leading coefficient in $x^{2n} - 2a^n x^n \cos np + a^{2n}$ is 1, we have the identity

$$x^{2n} - 2a^n x^n \cos np + a^{2n} = \prod_{0}^{n-1} [x^2 - 2ax\cos(p + 2r\pi/n) + a^2].$$

Results of §7·5 and §7·6 may be derived from this identity by putting $np=0$ and π respectively in it.

7·71. The following geometrical property is immediately obtained from the result of § 7·7. It is due to De Moivre.

D_1, D_2, D_3,... D_n are the vertices of a regular polygon of n sides inscribed in a circle, centre C and radius x and R is any pt. such that $CR=a$ and is inclined to any one of the radii CD_1, CD_2,... at an angle ϕ. Prove that

$$RD_1^2,\ RD_2^2,\ RD_3^2 \ldots RD_n^2 = x^{2n}-2a^n x^n \cos n\phi + a^{2n}.$$

Suppose $\angle RCD_1=\phi$. Now D_1, D_2,... D_n, are the vertices of a regular polygon of n sides; therefore each of the arcs D_1D_2, D_2D_3, D_3D_4...subtend an angle equal to b at the centre C, nb being equal to 2π. Hence RD_1, RD_2, RD_3,...subtend angles p, $p+b$, $p+2b$,... $p+(n-1)b$ at the centre. Now from the $\triangle RCD_1$, we have

$$RD_1^2 = CR^2+CD_1^2-2CR.\ CD_1 \cos\angle RCD_1 = x^2-2ax\cos p+a^2.$$

Smilarly from $\triangle s$ RCD_2, RCD_3...we have

$$RD_2^2 = x^2-2ax\cos(p+b)+a^2$$
$$RD_3^2 = x^2-2ax\cos(p+2b)+a^2,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$

Hence $RD_1^2.\ RD_2^2.RD_3^2 \ldots RD_n^2$ is equal to

$$\prod_{0}^{n-1}\left[x^2-2ax\cos(p+rb)+a^2\right] \quad \text{or} \quad x^{2n}-2a^n x^n\cos np+a^{2n}.$$

This result obviously includes as particular cases, those of § 7·51, and § 7·61. [The student may work out the suggestion.]

7·72. *Prove that if $na=2\pi$*

$$\cos nx-\cos ny = 2^{n-1}[\cos x-\cos y]\,[\cos x-\cos(a+y)]\,[\cos x-\cos(2a+y)]$$
$$\ldots\,[\cos x-\cos(n-1a+y)].$$

A proof of this result has been given already (§ 7·2, cor.2). It can however be easily deduced from the identity of § 7·7, by writing $\cos\frac{1}{2}x+i\sin\frac{1}{2}x$ for x, $\cos\frac{1}{2}x-i\sin\frac{1}{2}x$ for a and y for p; for then x^2+a^2 reduces to $2\cos x$, ax to 1 and $x^{2n}+a^{2n}$, to $2\cos nx$.

Example 1. Find the factors of x^6+x^3+1.

We have $x^6+x^3+1=x^6-2x^3\cos\frac{2}{3}\pi+1$, hence by § 7·7. we have

$$x^6+x^3+1=(x^2-2x\cos\tfrac{2}{9}\pi+1)(x^2-2x\cos\tfrac{8}{9}\pi+1)(x^2-2x\cos\tfrac{14}{9}\pi+1).$$

Example 2. Show that

$$\prod^{m}\cos(4r-1)a=(-1)^{\frac{1}{2}m}2^{-\frac{1}{2}\overline{2m-1}} \text{ if } m \text{ be even}$$

$$=(-1)^{\frac{1}{2}\overline{m+1}}2^{-\frac{1}{2}\overline{2m-1}} \text{ if } m \text{ be odd}$$

$4ma$ being equal to π. [*B. U. 1922 Hons.* II.]

Writing π, 3π, $2m$, and $4a$ for $2x$, $4my$, n, a respy., in the identity of § 7·72, we have

$$\cos m\pi=2^{2m-1}\cos 3a.\ \cos 7a.\ \cos 11a\ldots\cos(8m-5)a.\ \cos(8m-1)a. \quad (1)$$

Now $\cos(4m+3)a=-\cos 3a$, $\cos(4m+7)a=-\cos 7a$..... etc.

Making these substitutions in (1), we have

$$2^{2m-1}\cos^2 3a.\ \cos^2 7a\cos^2 11a\ldots\cos^2(4m-1)a=(-1)^m\cos m\pi=1 \quad (2)$$

Taking square roots of both sides of (2), we get

$$\cos 3a.\ \cos 7a.\ \cos 11a\ldots\cos(4m-1)a=\pm 2^{-\frac{1}{2}\overline{2m-1}}.$$

Now if m is even, $\cos(2m+3)a$, $\cos(2m+7)a$, $\cos(2m+11)a$, ...$\cos(4m-1)a$ are negative, for they are cosines of angles lying between $\frac{1}{2}\pi$ and π. Hence in this case we have

$$\cos 3a.\ \cos 7a.\ \cos 15a\ldots\cos(4m-1)a.=(-1)^{\frac{1}{2}m}2^{-\frac{1}{2}\overline{2m-1}}.$$

If m is odd, cosines of the angles $(2m+1)a$, $(2m+5)a$, $(2m+9)a$,...$(4m-1)a$, are negative similarly. Therefore when m is odd, we have

$$\cos 3a.\ \cos 7a.\ \cos 11a\ldots\cos(4m-1)a=(-1)^{\frac{1}{2}(m+1)}.2^{-\frac{1}{2}(2m-1)}.$$

Example 3. Prove that if $na=2\pi$,

$$2^{n-2}\sin x.\ \sin(x+a).\ \sin(x+2a)\ldots\sin[x+(n-1)a]$$
$$=\sin\tfrac{1}{2}nx.\ \sin\tfrac{1}{2}n(x+\pi).$$

$ABC\ldots$ is a regular polygon of n sides inscribed in a circle of radius unity and centre O. A pt. K is taken within it so that $OK=b$, and angle $\angle KOA=x$; shew that the product of the distances from the centre, of lines KA KB, $KC\ldots$ is equal to

$$\frac{b^n\sin\frac{1}{2}nx.\ \sin\frac{1}{2}n(x+\pi)}{2^{n-2}\sqrt{(b^{2n}-2b^n\cos nx+1)}}$$ [*B. U. 1916 Hons* II.]

Writing $\frac{1}{2}\pi$ for x and $x+\frac{1}{2}\pi$ for y, in the identity of §7·72, we have

$$\cos\tfrac{1}{2}n\pi-\cos n(x+\tfrac{1}{2}\pi)=2^{n-1}\sin x\sin(x+a)\ldots\ \sin[x+(n-1)a],$$

which immediately reduces to the first part of the question.

Denote the lengths of perpendiculars from the centre to KA, KB, KC,... by p_1, p_2,...p_r,...Then by considering the area of the $\triangle\ KOA$ in two ways, we have

$$2\triangle\ KOA=p_1.KA=OK.\ OA\sin x=b\sin x.$$

Similarly from $\triangle s$ KOB, KOC,... we have

$$p_2.\ KB=b\ \sin(x+a);\ p_3\ KC=b\ \sin(x+2a);\cdots\ .$$

$$\therefore\ p_1\ p_2\ldots p_n\ KA.\ KB.KC\ldots=b^n\sin x.\ \sin(x+a)\ldots\sin[x+(n-1)a].$$

$$=2^{-n+2}\,b^n\sin\tfrac{1}{2}nx.\ \sin\tfrac{1}{2}n(x+\pi) \qquad (1)$$

Now by §7·71, we have

$$KA,\ KB,\ KC\ldots\ =\sqrt{(b^{2n}-2b^n\cos nx+1)}.$$

Making this substitution in (1), we obtain the second part.

Example 4. An oblique conical surface stands on a circular base in which a regular polygon of n sides is inscribed. One side of the polygon is bisected at rt. angles by the diameter of the circle which passes through the longest and shortest generators of the cone, of lengths l_1 and l_2 respectively. Prove that the products of distances of the vertex of the cone from the angular points of the polygon is $[\frac{1}{2}(l_1+l_2)]^n+[\frac{1}{2}(l_1-l_2)]^n$. [*B. U. 1920 Hons. II.*]

Let O be the vertex and X the centre of the circle, E_1, E_2, E_3,...E_n the vertices of the inscribed polygon, OP, OQ, the longest and shortest gererators and PQ, the diameter bisecting at rt. angles the side $E_n\ E_1$. Draw the perpendicular $O\ N$ to the plane of the circle. It is evident from the symmetry of the figure that N must lie in the diameter QP.

Now $\angle\ O\ N\ E_r=\pi/2;\ \therefore\ O\ E_r^2=\ O\ N^2+N\ E_r^2$

Again $\angle\ N\ X\ E_r=(2r-1)\pi/n;\ \ \therefore$ from $\triangle\ NXE_r$, we have

$$NE_r^2=XE_r^2+XN^2-2XE_r.\ XN\cos\,[(2r-1)\pi/n].$$

$$\therefore\ \ OE_r^2=ON^2+\lambda E_r^2+XN^2-2X\ E_r.\ XN\cos[(2r-1)\,\pi/n].$$

Further $\angle\ ONX=\pi/2;\ \ \ \therefore\ \ \ $ from $\triangle\ OXN$, $OX^2=ON^2+NX^2$

and hence

$$OE_r^2 = OX^2 + XQ^2 - 2OX\ XN \cos[(2r-1)\pi/n].$$

Now since $QX = PX$, from $\triangle OQP$, we have

$$OX^2 + XQ^2 = \tfrac{1}{2}(OP^2 + OQ^2) = \tfrac{1}{2}(l_1^2 + l_2^2).$$

also $$4QX.\ XN = QN^2 - NP^2 = OQ^2 - OP^2 = l_1^2 - l_2^2$$

Hence we have

$$OE_r^2 = \tfrac{1}{2}(l_1^2 + l_2^2) - \tfrac{1}{2}(l_1^2 - l_2^2)\cos[(2r-1)\pi/n]$$
$$= [\tfrac{1}{2}(l_1 + l_2)]^2 - 2[\tfrac{1}{2}(l_1 + l_2)][\tfrac{1}{2}(l_1 - l_2)]\cos[(2r-1)\pi/n] + [\tfrac{1}{2}(l_1 - l_2)]^2.$$

By writing $r = 1, 2, 3, \ldots n$, we obtain similar expressions for OE_1^2, OE_2^2.....Multiplying them, and using the result of § 7·61, we have the required result at once.

Example 5. Prove that, if n is odd

$$(x+1)^n - (x-1)^n = 2n(x^2 + \tan^2 a)(x^2 + \tan^2 3a)(x^2 + \tan^2 5a)\ldots$$

to n factors $2na$ being equal to π.

If w represents any one of the nth roots of unity, $\cos ra + i\sin ra$ say, a root of $(x+1)^n = (x-1)^n$ is obviously given by

$$(x+1) = (x-1)w,$$

i. e. by $x = -(1+w)/(1-w) = -i\cot 2ra = -i\tan(n-2r)a$.

The equation must also be satisfied by the conjugate root $i\tan(n-2r)a$. The corresponding factor of $(x+1)^n - (x-1)^n$ is

$$x^2 + \tan^2(n-2r)a.$$

Writing $r = 1, 2, 3, \ldots \tfrac{1}{2}(n-1)$, we have $\tfrac{1}{2}(n-1)$ quadratic factors

$$x^2 + \tan^2 a,\ x^2 + \tan^2 3a,\ x^2 + \tan^2 5a, \ldots\ldots$$

Now the coefficient of x^{n-1} in $(x+1)^n - (x-1)^n$ is $2n$ and hence we get the required result.

Examples. VII. e.

Resolve the following polynomials into linear and quadratic factors:

1. $x^4 + 2x^2\cos 2a + 1$.
2. $x^4 + 2x^2\sin 2a + 1$.
2. $x^6 + x^3\sqrt{2} + 1$.
4. $x^6 - 2x^3\sin 3a + 1$.
5. $x^8 - 17x^4 + 16$.
6. $x^8 + 15x^4 - 16$.
7. $2x^6 - 1 - i\sqrt{3}$.
8. $x^6 + 2ix^3\cos 3\beta - 1$
9. Prove that, if $a = \cos\frac{1}{8}\pi + i\sin\frac{1}{8}\pi$ $x^4 - i = (x^2 - a^2)(x^2 - a^{10})$.
10. Prove that if $a = \cos\frac{1}{12}\pi + i\sin\frac{1}{12}\pi$

$$x^6 - i = (x^2 - a^2)(x^2 - a^{10})(x^2 - a^{18}).$$

7·8. The following example illustrates an interesting application of the nth roots of unity. Consider the determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ a_3 & a_1 & a_2 \\ a_2 & a_3 & a_1 \end{vmatrix}$$

The elements of the second and the third rows are the two cyclic permutations a_3, a_1, a_2, and a_2, a_3, a_1, of the elements of the 1st row a_1, a_2, a_3. For this reason such determinants are called circulants. They can be easily expanded into the form of a product of linear factors.

Multiply the 2nd and the 3rd columns by w, w^2 and add these **circulants** to the 1st column: w being any cube root of unity. The determinant then reduces to

$$\begin{vmatrix} a_1+a_2w+a_3w^2 & a_2 & a_3 \\ a_3+a_1w+a_2w^2 & a_1 & a_2 \\ a_2+a_3w+a_2w^2 & a_3 & a_1 \end{vmatrix}.$$

Now it is easy to see that the elements of the 1st column of the transformed determinant can be written thus:

$$a_1+a_2w+a_3w^2,\ w(a_1+a_2w+a_3w^2),\ w^2(a_1+a_2w+a_3w^2).$$

Hence $a_1+a_2w+a_3w^2$ is a factor of the determinant. Thus corresponding to the three cube roots of unity, the determinant has the three factors $a_1+a_2+a_3$, $a_1+a_2w+a_3w^2$, $a_1+a_2w^2+a_3w$.

Now observing that the coefficient of a_1^3 in the determinant is unity, we have the identity

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ a_3 & a_1 & a_2 \\ a_2 & a_3 & a_1 \end{vmatrix} = (a_1+a_2+a_3)(a_1+a_2w+a_3w^2)(a_1+a_2w^2+a_3w) \qquad (\S\ 5{\cdot}411)$$

The method can be easily generalised. When the determinant is a *circulant* of the nth order, we employ the nth roots of unity.

Examples. VII. f.

1. Prove that $\begin{vmatrix} 1 & x & 0 & 0 \\ 0 & 1 & x & 0 \\ 0 & 0 & 1 & x \\ x & 0 & 0 & 1 \end{vmatrix} = 1-x^4.$

2. Prove that $\begin{vmatrix} a & b & b & b \\ b & a & b & b \\ b & b & a & b \\ b & b & b & a \end{vmatrix} = (a+3b)(a-b)^3.$

3, Express as a polynomial

$$\begin{vmatrix} 1 & x & x^2 & 0 \\ 0 & 1 & x & x^2 \\ x^2 & 0 & 1 & x \\ x & x^2 & 0 & 1 \end{vmatrix}$$

[*B. U. 1925 Hons.* I.]

4. Prove that $\begin{vmatrix} 1 & 3 & 3 & 0 \\ 0 & 1 & 3 & 3 \\ 3 & 0 & 1 & 3 \\ 3 & 3 & 0 & 1 \end{vmatrix} = 91.$

5. Apply the method of this article to prove that

$$\begin{vmatrix} a+b & a+2b & a+3b & a+4b \\ a+2b & a+3b & a+4b & a+b \\ a+3b & a+4b & a+b & a+2b \\ a+4b & a+b & a+2b & a+3b \end{vmatrix} = 32(2a+5b)b^3$$

[*B. U. 1916 Hons.* I.]

[Interchange second and fourth rows, it then reduces to the form of a circulant. See § 2·64, illustrative example 6, for an alternative method].

6. Prove that $\begin{vmatrix} 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 1 & 0 & 0 & 1 & 2 \\ 2 & 1 & 0 & 0 & 1 \end{vmatrix} = 2^2.$

Examples. VII. g.

Resolve into factors

1. $x^n+x^{n-1}+x^{n-2}+\ldots+x+1$,
2. $x^n-x^{n-1}+x^{n-2}-x^{n-3}+\ldots+(-1)^n$.

3. If $4na=\pi$ shew that $\cot a, -\cot 3a, \cot 5a, -\cot 7a, \ldots$ $(-1)^{n-1}\cot(2n-1)a$ are the roots of

$$x^n - {}_nC_1 x^{n-1} - {}_nC_2 x^{n-2} + {}_nC_3 x^{n-3} + \ldots = 0.$$

Dednce that

$\cot a.\ \mathrm{cosec}^2 a - \cot 3a.\ \mathrm{cosec}^2 3a + \ldots$ to n terms $= 2n^3$.

[*B. U. 1921 Problems* and *B. U. 1915 Hons II.*]

[Write $\frac{1}{4}\pi$ for na and $1/x$ for x in the equation of §7·3]

4. When n is an odd integer, prove that

$$\cot^4 2a + \cot^4 4a + \cot^4 6a + \ldots + \cot^4(n-1)a$$
$$= (n-1)(n-2)(n^2+3n-13) \div 90$$

where $2na=\pi$. [*B. U. 1917 Hons. II.*]

5. Show that

$$\sum_1^{n-1} \mathrm{cosec}^2\left(a + \frac{r\pi}{n}\right) = n^2 \mathrm{cosec}^2 na - \mathrm{cosec}^2 a.$$

[*B. U. 1925 Hons. II.*]

6. Prove that if $2na=\pi$,

$$(1+x)^n + (1-x)^n = 2(1+x^2\cot^2 a)(1+x^2\cot^2 3a)(1+x^2\cot^2 5a) \ \ldots\ldots$$

the numbers of factors on the right being $\frac{1}{2}n$ or $\frac{1}{2}(n-1)$ according as n is even or odd-

$$\left.\begin{aligned} &(1+x)^n = (1-x)^n[\cos(2r-1)\pi + i\sin(2r-1)\pi], \\ &\therefore 1+x = (1-x)[\cos(2r-1)\pi/n + i\sin(2r-1)\pi/n]. \end{aligned}\right\}$$

7. Prove that if $na=\pi$,

$$(1+x)^n - (1-x)^n = 2nx(1+x^2\cot^2 a)(1+x^2\cot^2 2a)(1+x^2\cot^2 3a)$$

the number of brackets on the right, being $\frac{1}{2}(n-2)$ or $\frac{1}{2}(n-1)$ according as n is even or odd.

8. When n is even and $2na=\pi$, prove that

i. $2^{\frac{1}{2}n-1}\sin a.\ \sin 3\cdot \sin 5a \ldots \sin(n-1)a = 1$;

ii. $2^{\frac{1}{2}n-1}\cos a.\ \cos 3a.\ \cos 5a \ldots \cos(n-1)a = 1$;

iii. $2^{\frac{1}{2}n-1}\sin 2a.\ \sin 4a.\ \sin 6a \ldots \sin(n-2)a = \sqrt{n}$;

iv. $2^{\frac{1}{2}n-1}\cos 2x\ \cos 4x.\ \cos 6a \ldots \cos(n-2)a = \sqrt{n}$;

v. $\tan a.\ \tan 3a.\ \tan 5a \ldots \tan(n-1)a = 1$;

vi. $\tan 2a.\ \tan 4a.\ \tan 6a \ldots \tan(n-2)a = 1$;

9. When n is odd and $2na=\pi$, prove that

i. $2^{-\frac{1}{2}\overline{n-1}}=\sin a.\ \sin 3a.\ \sin 5a\ldots\sin(n-2)a$

ii. $=\cos 2a.\ \cos 4a\ \cos 6a\ldots\cos(n-1)a$;

iii. $2^{-\frac{1}{2}\overline{n-1}}\sqrt{n}=\sin 2a.\ \sin 4a.\ \sin 6a\ldots\sin(n-1)a$;

iv. $=\cos a.\ \cos 3a.\ \cos 5a\ldots\cos(n-2)a$;

v. $\sqrt{n}=\tan 2a.\ \tan 4a.\ \tan 6a\ldots\tan(n-1)a$

vi. $=\cot a.\ \cot 3a.\ \cot 5a\ldots\cot(n-2)a$.

10. If n is an odd integer and $2na$ is equal to π, prove that

i $3^n-1=2n(4+\cot^2 2a)(4+\cot^2 4a)(4+\cot^2 6a)\ldots$

$[4+\cot^2(n-1)a]$;

ii. $3^n+1=4(4+\tan^2 2a)(4+\tan^2 4a)(4+\tan^2 6a)\ldots$

$[4+\tan^2(n-1)a]$.

[Put $2x=1$ examples 6, 7, above]. [*B. U. 1922 Problems*]

11. A regular polygon $ABCD\ldots$ of n sides is inscribed in a circle of radius a. From the mid point of the arc AB, perpendiculars are drawn to the sides of the polygon. Prove that the product of the n perpendiculars is $4(a/2)^n$. [*B. U. 1926 Hons. II.*]

12. Prove that if $n\pi=b$

$$\tan^{-1}\frac{x\sin y}{a-x\cos y}+\tan^{-1}\frac{x\sin(y+b)}{a-x\cos(y+b)}+\tan^{-1}\frac{x\sin(y+2b)}{a-x\cos(y+2b)}$$

$$+\tan^{-1}\frac{x\sin(y+3b)}{a-x\cos(y+3b)}+\ldots+2n\text{ terms}=\tan^{-1}\frac{x^{2n}\sin 2ny}{a^{2n}-x^{2n}\cos 2ny}$$

[*B. U. 1921 Hons. II*]

[We have $a^{2n}-x^{2n}\cos 2ny+ix^{2n}\sin 2ny=a^{2n}-x^{2n}(\cos y-i\sin y)^{2n}$

$=[a-x(\cos y-i\sin y)][a-x(\cos y+b-i\sin y+b)]\ldots 2n.$ factors. Now use the result of example 16 IV, c]

13. A series of radii divides the circumference of a circle into $2n$ equal parts; shew that the product of the perpendiculars let fall from any point of the circumference on n successive radii is equal to $r^n\sin nq/2^{n-1}$ where r is the radius, and q the angle between the radius to the given point and one of the extreme radii

[*B. U. 1918 Hons. II.*]

14. Prove that $(1-x^4)^3$ is equal to

$$\begin{vmatrix} 1 & x & x^2 & x^3 \\ x^3 & 1 & x & x^2 \\ x^2 & x^3 & 1 & x \\ x & x^2 & x^3 & 1 \end{vmatrix}.$$

C. H. S. 1907

15. If $a, b, c, \ldots$ are the roots of

$$f(x) \equiv x^m + px^{m-1} + qx^{m-2} + \ldots = 0,$$

and w is a special nth root of unity, prove that

$$f(x).f(xw).f(xw^2)\ldots f(xw^{n-1}) \equiv (-1)^{mn+n}(x^n-a^n)(x^n-b^n)(x^n-c^n)\ldots$$

16. Shew that $[\cos(2m+1)x+1] \div (\cos x+1)$ can be expressed, as a square of a polynomial in $\cos x$. *e. g.*

$$2\cos 11x+2 = (y+2)(y^5-y^4-4y^3+3y^2+3y-1)^2$$

where $y = 2\cos x$.

17. Prove the following two identities

I. $x^5-5abx^3+5a^2b^2x+a^5+b^5 = (x+a+b)$
$\times[x^2+2(a+b)x\cos 72^0+a^2+b^2+2ab\cos 144^0]$
$\times[x^2+2(a+b)x\cos 144^0+a^2+b^2+2ab\cos 72^0]$.

II. $x^6-6abx^4+9a^2b^2x^2-(a^3+b^3)^2 = [x^2-(a+b)^2]$
$\times[x^2+(a+b)x+a^2-ab+b^2]\times[x^2-(a+b)x+a^2-ab+b^2]$.

18. Prove that the equation $k\cot nx+\cot(a-x)=0$ has $n+1$ solutions, no two of which differ by a multiple of π.

If these solutions are $p_0, p_1, p_2, \ldots p_n$, prove that

$$\sum_0^n \cot(p_r-a) = (k-n)\cot na. \qquad [B.\ U.\ 1927\ Hons.\ \text{II}].$$

[Consider the equation $k\cot n(x+a)-\cot x=0$ obtained by writing $x+a$ for x in the given equation].

19. Prove that if $nb = 2\pi$

$[1+\sin 2a\cos x][1+\sin 2a\cos(x+b)][1+\sin 2a\cos(x+2b)]$.

...to n factors $= [\sin^{2n}a-(-1)^n 2\sin^n a\cos^n a\cos nx+\cos^{2n}a]$.

Radii vectors $SP_1, SP_2, \ldots SP_n$ are drawn from the focus, S of an ellipse meeting it in $P_1, P_2, \ldots, P_n$. If the angle between any two consecutive radii vectors is $2\pi/n$, prove that

$$2^n b^{2n} = SP_1\ SP_2.\ SP_3\ldots SP_n$$
$$\times[(a+b)^n-2(a^2-b^2)^{\frac{1}{2}n}(-1)^n\cos nx+(a-b)^n].$$ where a, b are the semi axes of the ell

CHAPTER VIII

PARTIAL FRACTIONS

8·1. We know how to express a sum of fractions such as $1/(x-1)+1/(x+1)$ in the form of a single fraction $2x/(x^2-1)$. In many problems however, it is convenient to exhibit a rational function in the form of a sum of *partial fractions*–fractions whose denominators are of lower degrees than that of the given fraction. This chapter is devoted to the study of methods of effecting this transformation.

8·2. The fundamental theorem of the chapter depends on the following lemma:–

If two polynomials C and D are prime to each other–they have no common factor–*it is possible to find two other polynomials L and M such that $L.C+M.D=1$.*

Suppose C and D are of the mth and nth degrees, m being greater than n. Apply the process of finding the H. C. F. to these polynomials and denote the corresponding remainders, that occur in the process, by $D_1, D_2, D_3 \ldots D_n$ and $R_1, R_2, R_3 \ldots R_n$. We have the following obvious identities:–

$$C = D.\ D_1 + R_1; \quad \therefore \quad R_1 = C - D_1.\ D; \tag{1}$$

$$D = R_1.\ D_2 + R_2; \quad \therefore \quad R_2 = D - R_1.\ D_2; \tag{2}$$

$$R_1 = R_2.\ D_3 + R_3; \quad \therefore \quad R_3 = R_1 - R_2.\ D_3; \tag{3}$$

$$R_2 = R_3.\ D_4 + R_4; \quad \therefore \quad R_4 = R_2 - R_3.\ D_4; \tag{4}$$

.....

$$\text{and} \quad R_{n-2} = R_{n-1}.\ D_n + R_n \quad \therefore \quad R_n = R_{n-2} - R_{n-1} D_n. \tag{n}$$

From (1) we have $R_1 = C - D_1.\ D$, (1′)

Substituting this value of R_1 in (2), we get

$$R_2 = D - D_2(C - D_1.\ D) = -D_2 C + (1 + D_1.\ D_2) D. \tag{2′}$$

Substituting the values of R_1, and R_2 from (1′) and (2′) in (3) we have similarly

$$R_3 = (1 + D_2.\ D_3) C + (-D_1 - D_3 - D_1.\ D_2.\ D_3) D. \tag{3′}$$

In this way, we can express each of the remainders R_4, R_5,... and finally R_n as a linear homogeneous function of C and D. We thus have

$$R_n = L'C + M'D, \qquad (n')$$

where L' and M' are clearly some polynomials.

Now since C and D have by hypothesis, no common factor, the last remainder R_n must be a constant different from zero. Dividing the identity (n') by R_n and writing L and M for L'/R_n and M'/R_n, we have the result in the required form.

8·3. *If any two polynomials C and D be prime to each other, then any fraction $R/(C.\ D)$ can be thrown into the form*

$$R/(C.\ D) = N + A/C + B/D.$$

where N, A, B,....., are polynomials, A and B being of lower degrees than C and D respectively.

Since C and D are prime to each other, we know, from § 8·2, that two polynomials L and M exist such that

$$1 = L.\ C + M.\ D. \qquad (1)$$

Multiplying both sides of the identity (1) by $R/(C.\ D.)$ we have

$$R/(C.\ D) = R.\ M/C + R.\ L/D. \qquad (2)$$

If $R.\ M$ be of a higher degree than C, divide $R.\ M$ by C and let the quotient and the remainder be E and A, so that we have

$$R.\ M/C = E + A/C. \qquad (3)$$

Similary suppose $R.\ L/D$ reduces to $F + B/D$. (4)

We thus have, from (2), (3), (4) .

$$R/(C.\ D) = E + F + A/C + B/D. \qquad (5)$$

Now writing N for $E+F$, we have the identity in the required form.

Remarks:-It should be noticed that the degree of N is equal to the degree of R minus the degree of $C.\ D$. For, multiplyiug (5) by $C.\ D$ we have

$$R = N.\ C.\ D + A.\ D + B.\ C. \qquad (6)$$

Now A and B are of lower degrees than C and D respectively, therefore $A.\ D$ and $B.\ C$ are of lower degrees of than $C.\ D$. Hence the degree of the right hand member of (6) is the same as that of

N. C. D. Therefore R is of the same degree as *N. C. D.* Hence the degree of N is equal to the degree of R minus the degree of *C. D.* In particular, if the given fraction is proper, the polynomial part N of the right hand expression of (6) must identically vanish.

8·31. *Polynomials P, Q, R,...are prime to each other and are factors of the denominator of a fraction. Prove that the fraction can be put into the form*

$$N+\frac{A}{P}+\frac{B}{Q}+\frac{C}{R}+\ldots,$$

A, B, C...being polynomials of lower degrees than P, Q, R,...respectively.

The theorem can be proved by the repeated application of § 8·3. Since, P, Q, R,...are prime to each other, P and the product $Q.R$....are prime to each other. Hence by §8·3, the fraction can be put into the form $N+\frac{A}{P}+\frac{B'}{QR\ldots}$. (1)

where A and B' are polynomials of lower degrees than P and $Q.R$...respectively. Applying the same argument to the fraction $B'/Q.R\ldots$, we see that it can be put into the form $\frac{B}{Q}+\frac{C'}{R\ldots}$ (2)

Using (1) and (2) we find that the given fraction can be put into the form

$$N+\frac{A}{P}+\frac{B}{Q}+\frac{C'}{R\ldots}$$

Repeating the process sufficient number of times, the required result can be easily obtained.

Example I. Resolve into partial fractions

$$\frac{x^2-x}{(x^3+x+1)(x^2+x+1)}.$$

When x^3+x+1 is divided by x^2+x+1, the quotient and the remainder are $x-1$, and $x+2$ respectively. Hence

$$x^3+x+1=(x-1)(x^2+x+1)+x+2. \quad (1)$$

Similarly when x^2+x+1 is divided by $x+2$, the quotient and the remainder are $x-1$, and 3;

$$x^2+x+1=(x+2)(x-1)+3. \quad (2)$$

From (1) and (2), we get

$$x^2+x+1=(x-1)[(x^3+x+1)-(x-1)(x^2+x+1)]+3;$$

$$\therefore\ 3=(x^2+x+1)[(x-1)^2+1]-(x-1)(x^2+x+1);$$

$$\therefore\ 3(x^2-x)=(x^2-x)(x^2-2x+2)(x^2+x+1)-(x^2-x)(x-1)(x^3+x+1).$$

Dividing this result by $(x^3+x+1)(x^2+x+1)$, we get

$$\frac{3(x^2-x)}{(x^2+x+1)(x^3+x+1)}=\frac{(x^2-x)(x^2-2x+2)}{x^3+x+1}-\frac{(x^2-x)(x-1)}{x^2+x+1}. \quad (3)$$

Now

$$\frac{(x^2-x)(x^2-2x+2)}{x^3+x+1}=x-3+\frac{3x^2+3}{x^3+x+1}$$

and

$$\frac{(x^2-x)(x-1)}{x^2+x+1}=x-3+\frac{3x+3}{x^2+x+1}.$$

Hence (3) reduces to

$$\frac{x^2-x}{(x^2+x+1)(x^3+x+1)}=\frac{x^2+1}{x^3+x+1}-\frac{x+1}{x^2+x+1}.$$

Otherwise:—We know from §8·3, that the fraction can be written thus:

$$\frac{x^2-x}{(x^2+x+1)(x^3+x+1)}=\frac{lx+m}{x^2+x+1}+\frac{l'x^2+m'x+n'}{x^3+x+1} \quad (4)$$

where l, m, l', m', n' are constants. Multiplying the identity (4) by $(x^2+x+1)(x^3+x+1)$, we get

$$x^2-x=(lx+m)(x^3+x+1)+(l'x^2+m'x+n')(x^2+x+1). \quad (5)$$

Equating the coefficients of the varions powers of x on both sides of (5), we get the equations

$$l+l'=0;\ m+l'+m'=0;\ 1=l+l'+m'+n';$$

$$-1=l+m+m'+n' \text{ and } 0=m+n'.$$

From these simultaneons equations, we have

$$m'=0 \text{ and } -1=l=m=-n'=-l'$$

Hence (4) reduces to

$$\frac{x^2-x}{(x^2+x+1)(x^3+x+1)}=\frac{x^2+1}{x^3+x+1}-\frac{x+1}{x^2+x+1}.$$

Both the methods suggested in this example are of universal application. Much simplification can, however, be introduced by employing certain devices. In the following articles, we explain the principles underlying them.

8·32. The case of **a single non-repeated binomial factor** in the denominator of a fraction is of frequent occurrence. Let the factor be $x-a$, so that the given fraction is of the form $f(x)/P(x)$ where $P(x) \equiv (x-a)Q(x)$; $f(x)$, $P(x)$, $Q(x)$ denoting certain polynomials.

Now by § 8·3, the fraction can be written in the form

$$\frac{f(x)}{P(x)} = \frac{A}{x-a} + \frac{F(x)}{Q(x)} \qquad (1)$$

where A is independent of x and $F(x)$ is a polynomial. Multiplying both sides of (1) by $P(x)$, we have

$$f(x) = AQ(x) + (x-a)F(x).$$

Putting $x=a$ in the identity we get

$$f(a) = A.\ Q(a) \qquad \text{or} \qquad A = f(a)/Q(a). \qquad (3)$$

We thus deduce the rule:-*The partial fraction corresponding to a single non-repeated binomial factor* $x-a$, is $A \div (x-a)$ *where* A *is obtained by putting* a *for* x *in the remaining body of the fraction.*

To apply this rule, it is necessary to express the denominator of the given fraction in the form of a product of factors $(x-a)Q(x)$. A suitable formula in cases where the denominator cannot be conveniently factorised is derived thus:-

Differentiating the identity $P(x) = (x-a)Q(x)$, we have

$$P'(x) = Q(x) + (x-a)Q'(x).$$

Putting $x=a$ in this, we have $P'(a) = Q(a)$.

Substituting this value of $Q(a)$ in the formula (3), we have

$$A = f(a) \div P'(a). \qquad (4')$$

Remarks:-It should be observed that in the above argument we have assumed nothing regarding either the degree of the numerator or the nature of the number a; hence the numerator may be of a higher degree than the denominator and a may be real or complex.

Example 1. Resolve into partial fractions

$$\frac{x^3}{(x-2)(x-1)(x+1)}.$$

Applying the formula (3) of § 8·32, the partial fraction corresponding to $(x-2)$ is

$$\frac{1}{x-2} \cdot \frac{2^3}{(2-1)(2+1)} \qquad \text{or} \qquad \frac{8}{3} \cdot \frac{1}{x-2}.$$

Similarly partial fractions corresponding to $x+1$ and $x-1$ are

$$-\tfrac{1}{6}\,\frac{1}{x+1} \text{ and } -\tfrac{1}{2}\,\frac{1}{x-1}.$$

Further, if x^3 is divided by $(x-2)(x+1)(x-1)$, the quotient is clearly 1; hence the polynomial part consists of 1.

$$\therefore \frac{x^3}{(x-2)(x-1)(x+1)} = 1 + \tfrac{8}{3}\cdot\frac{1}{x-2} - \tfrac{1}{6}\cdot\frac{1}{x+1} - \tfrac{1}{2}\,\frac{1}{x-1}.$$

Example 2. Shew that

$$\frac{{}_nC_0}{x} - \frac{{}_nC_1}{x+1} + \frac{{}_nC_2}{x+2} - \ldots + \frac{(-1)^n\,{}_nC_n}{x+n} = \frac{n!}{x(x+1)(x+2)\ldots(x+n)}$$

[*B. U. 1918 Hons.* I.]

To establish the identity, we resolve the right hand fraction into partial fractions. Applying the formula (3) of § 8·32, the partial fraction corresponding to x is A_0/x where

$$A_0 = n!/1\cdot2\cdot3\ldots n = 1.$$

Similarly partial fractions corresponding to
$x+1$, $x+2$, $x+3$,...are $A_1/(x+1)$, $A_2/(x+2)$,...$A_r/(x+r)$...where

$$A_1 = n!/(-1).\ 1.2.3.4\ldots(-1+n) = -n/1 = (-1)^1\,{}_nC_1;$$
$$A_2 = n!/(-2)(-2+1)(-2+3)(-2+n) = (-1)^2 n!/2!(n-2)! = (-1)^2\,{}_nC_2;,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$A_r = n!/(-r)(-r+1)(-r+2)\ldots(-1)(-r+r+1)(-r+r+2)\ldots(-r+n)$$
$$= (-1)^r\,{}_nC_r;$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Otherwise. The result may also be obtained by integrating between 0 and 1, the obvious identity,

$$t^{x-1}(1-t)^n = t^{x-1} - {}_nC_1 t^x + {}_nC_2 t^{x+1} - {}_nC_3 t^{x+2} + \ldots + (-1)^n\,{}_nC_n t^{n+x-1}.$$

Example 3. Resolve into partial fractions $x^3/(x^4-1)$.

The roots of x^4-1 are clearly 1, −1, i, $-i$; hence the factors of x^4-1 are $x-1$, $x+1$, $x-i$, and $x+i$. Applying (3′) of § 8·32, the partial fraction corresponding to the factor $x-1$ is $A/(x-1)$ where A is obtained by putting 1 for x in $x^3/4x^3$ and is therefore equal to 1/4.

Similarly, partial fractions corresponding to $x+1$, $x-i$, $x+i$ are $1/4(x+1)$, $1/4(x-i)$, $1/4(x+i)$ respectively. Since the given fraction is a proper one, the polynomial part does not exist; hence we have

$$\frac{x^3}{x^4-1} = \tfrac{1}{4}\left\{\frac{1}{x-1}+\frac{1}{x+1}+\frac{1}{x-i}+\frac{1}{x+i}\right\}. \qquad (1)$$

Expressing the sum of the last two fractions as a single fraction, (1) transforms into

$$\frac{x^3}{x^4-1} = \tfrac{1}{4}\left\{\frac{1}{x-1}+\frac{1}{x+1}+\frac{2x}{x^2+1}\right\}.$$

N. B. *Example* 4. Resolve into partial fractions

$$1 \div [(1+x)^5+(1-x)^5].$$

If $10a=\pi$ and $w=\cos 2(2r-1)a+i\sin 2(2r-1)a$, $(r=1, 2, 3, 4,)$ then it is easily seen that a root of $(1+x)^5+(1-x)^5=0$ is given by

$$1+x=(1-x)w \quad \text{or} \quad x=i\tan(2r-1)a,$$

Now $(1+x)^5+(1-x)^5$ is an even function of x and is of the fourth degree; hence $-i\tan(2r-1)a$ must also be a root. Therefore $x^2+\tan^2 a$, and $x^2+\tan^2 3a$ must be the factors of the denominator. The corresponding partial fractions are, by §8·32, $A \div (x^2+\tan^2 a)$ and $B \div (x^2+\tan^2 3a)$ where A and B are obtained by putting $-\tan^2 a$ and $-\tan^2 3a$ for x^2 in

$$1 \div \frac{d}{d(x^2)}[(1+x)^5+(1-x)^5] \quad \text{or} \quad \tfrac{2}{5}x \div [(1+x)^4-(1-x)^4].$$

$$\therefore \quad A=\tfrac{2}{5}\,\frac{i\tan a}{(1+i\tan a)^4-(1-i\tan a)^4}=\tfrac{2}{5}\,\frac{\sin a.\cos^3 a}{2\sin 4a}$$

$$=\tfrac{1}{10}\cos 3a.\cos a.$$

Similarly $$B=-\tfrac{1}{10}\cos a\cos 3a.$$

Thus $$\frac{10\sec a.\sec 3a}{(1+x)^5+(1-x)^5}=\frac{1}{x^2+\tan^2 a}-\frac{1}{x^2+\tan^2 3a}.$$

Examples. VII. a.

Resolve the following into partial fractions:–

1. $\dfrac{x^2}{(x+1)(x+2)(x+3)}$.

2. $\dfrac{x^2}{(x-a)(x-b)(x-c)}$.

3. $\dfrac{x3}{(ax+1)(bx+1)(cx+1)}$.

4. $\dfrac{x^2}{x^4+1}$.

5. $\dfrac{x^2}{(x+1)(x^2+1)}$.

6. $\dfrac{x^4}{(x^2+a^2)(x^2+b^2)(x^2+c^2)}$.

7. $\dfrac{x^2}{1+x^2+x^4}$.

8. $\dfrac{x}{(x+1)^5-x^5-1}$.

(9) Prove the identity

$$\frac{x^5}{(x-a)(x-b)(x-c)(x-d)} = x+a+b+c+d+\sum \frac{a^5}{(x-a)(a-b)(a-c)(a-d)}.$$

Hence or otherwise, prove that

$$\frac{a^4}{(a-b)(a-c)(a-d)}+\frac{b^4}{(b-a)(b-c)(b-d)}+\frac{c^4}{(c-a)(c-b)(c-d)}$$
$$+\frac{d^4}{(d-a)(d-b)(d-c)} = a+b+c+d.$$

8.33. When a **repeated factor** of the form $(x-a)^r$ occurs in the denominator of a fraction, the corresponding partial fraction, by §8·3, can be put into the form $P(x)/(x-a)^r$ where $P(x)$ is a polynomial of the $r-1$ th degree at most. Now $P(x)$ can be put into the form (Read remarks on illustrative example 1, § 1·33.)

$$P(x) = p+q(x-a)+r(x-a)^2+\ldots+t(x-a)^{r-1}$$

where p, q, r,.....are constants. Hence the partial fraction $P(x)/(x-a)^r$ can be transformed into

$$\frac{P(x)}{(x-a)^r} = \frac{p}{(x-a)^r}+\frac{q}{(x-a)^{r-1}}+\frac{r}{(x-a)^{r-2}}+\ldots+\frac{t}{x-a}.$$

How the transformation is effected in practice, may be seen from the following examples.

Example 1. Resolve into partial fractions $8x/(x-1)^3(x+1)$.

Putting $x-1=y$ or $x=1+y$, the fraction reduces to

$$8(1+y)/y^3(2+y)$$

Now by actual division or by inspection, we have

$$8+8y = (2+y)(4+2y-y^2)+y^3. \qquad (1)$$

Dividing (1) by $y^3(2+y)$, we get

$$8(1+y)/y^3(2+y) = 4/y^3+2/y^2-1/y+1\div(2+y). \qquad (2)$$

Substituting the value of y in (2), we have

$$\frac{8x}{(x-1)^3(x+1)} = \frac{4}{(x-1)^3}+\frac{2}{(x-1)^2}-\frac{1}{(x-1)}+\frac{1}{x+1}.$$

Otherwise:- We know, from §8·33, that the fraction can be thrown into the form

$$\frac{8x}{(x-1)^3(x+1)} = \frac{a}{(x-1)^3}+\frac{b}{(x-1)^2}+\frac{c}{(x-1)}+\frac{d}{x+1}, \qquad (3)$$

where a, b, c d are constants. Multiplying by $(x-1)^3(x+1)$, we get

$$8x = a(x+1)+b(x-1)(x+1)+c(x-1)^2(x+1)+d(x-1)^3. \quad (4)$$

Putting $x=-1$ in (4), we have $d=1$.

Similarly putting $x=1$ in (4) we get $a=4$.

Differentiating the identity (4), with respect to x and then putting $x=1$, we have $8=a+2b$, $\therefore\ b=2$.

Differentiating (4) twice and then putting $x=1$, we get

$$0=2b+c(6-2),\ \therefore\ c=-1.$$

A third method. Constants a, b, c, d, may be evaluated by equating the coefficients of powers of x on both sides of the identity (4).

Example 2. Resolve into partial fractions $1\div(x^2-1)^3$.

Now $$2\div(1-x^2)=1/[1/(1-x)+1/(1+x)].$$

Hence
$$\frac{1}{(1-x^2)^3}=\frac{1}{8}\left(\frac{1}{1-x}+\frac{1}{1+x}\right)^3$$

$$=\frac{1}{8}\left\{\frac{1}{(1-x)^3}+\frac{3}{(1-x)^2(1+x)}+\frac{3}{(1-x)(1+x)^2}+\frac{1}{(1+x)^3}\right\}$$

$$=\frac{1}{8}\left\{\frac{1}{(1-x)^3}+\frac{3}{2}\,\frac{1}{1-x}\left[\frac{1}{1-x}+\frac{1}{1+x}\right]\right.$$

$$\left.+\frac{3}{2}\left[\frac{1}{1-x}+\frac{1}{1+x}\right]\frac{1}{1+x}+\frac{1}{(1+x)^3}\right\}$$

$$=\frac{1}{8}\left\{\frac{1}{(1-x)^3}+\frac{3}{2}\,\frac{1}{(1-x)^2}+\frac{3}{2}\,\frac{1}{1-x}+\frac{3}{2}\,\frac{1}{1+x}\right.$$

$$\left.+\frac{3}{2}\,\frac{1}{(1+x)^2}+\frac{1}{(1+x)^3}\right\}.$$

Example. 3. Resolve into partial fractions $(1-x)^n/(1-2x)^3$.

Putting $1-2x=y$, the fraction reduces to

$$(1+y)^n/2^n y^3=(1+{}_nC_1y+{}_nC_2y^2+\ldots+{}_nC_ny^n)\div 2^n y^3$$

$$=(1/y^3+{}_nC_1\div y^2+{}_nC_2\div y+{}_nC_3+{}_nC_4y+\ldots{}_nC_ny^{n-3})\div 2^n$$

Restoring the value of y, we have

$$\frac{(1-x)^n}{(1-2x)^3}=\frac{1}{2^n}\left\{\frac{1}{(1-2x)^3}+\frac{n}{(1-2x)^2}+\frac{n(n-1)}{2(1-2x)}+P\right\}$$

where P denotes a polynomial of the $(n-3)$ th degree.

Examples. VIII. b.

Resolve into partial partial fractions:–

1. $1 \div x^2(x^2+1)$.
2. $1 \div x(x+1)^5$.
3. $1 \div x^4(x+1)^3$.
4. $1 \div x^4(x+1)^4$.
5. $(x-1) \div x^2(x+1)^2$.
6. $1 \div x^2(x+1)^2(x+1)^2$.

8·34. Lastly, when a **power of a quadratic expression** $(x^2+bx+c)^n$ occurs as a factor in the denominator of a fraction, the corresponding partial fraction is, by §8·3, of the from

$$\frac{F(x)}{(x^2+bx+c)^n}$$

where $F(x)$ is a polynomial of the $(2n-1)$th degree at most. Now any polynomial like $F(x)$ can be put into the form *

$$(a_0+b_0x)+(a_1+b_1x)P^1+(a_2+b_2x)P^2+\ldots+(a_{n-1}+b_{n-1}x)P^{n-1},$$

where $P=x^2+bx+c$. Using this form of $F(x)$, the partial fraction can be expressed in a simpler form, as the sum of the n partial fractions

$$\frac{a_0+b_0x}{P^n}+\frac{a_1+b_1x}{P^{n-1}}+\frac{a_2+b_2x}{P^{n-2}}+\frac{a_3+b_3x}{P^{n-3}}+\ldots+\frac{a_{n-1}+b_{n-1}x}{P}.$$

* *N. B.* How the transformation is effected, is seen thus:– Divide $F(x)$ by P and suppose that the quotient and remainder are Q_0, R_0, where R_0 must necessarily be of the form a_0+b_0x. We thus have

$$F(x)=(a_0+b_0x)+Q_0P. \qquad (1)$$

If Q_0 is of a higher degree than one, divide it by P and suppose that the quotient and remainder are Q_1, R_1, where R_1 must be of the form a_1+b_1x. We thus have $Q_0=(a_1+b_1x)+Q_1.P$. Using this value of Q_0 in (1), we get

$$F(x)=(a_0+b_0x)+(a_1+b_1x)P+Q_1P^2. \qquad (2)$$

Applying this process repeatedly, we get the necessary transformation of $F(x)$.

Example 1. Resolve into partial fractions

$$\frac{x^4+4x^3+7}{(x-1)(x^2+1)^2}. \qquad [B.\ U.\ 1924\ Pass\ I.]$$

Now by §8·32, the partial fraction corresponding to $x-1$ is of the form $A \div (x-1)$, while by §8·34, the partial fractions corresponding to $(x^2+1)^2$, are of the form

$$(a_0+b_0x) \div (x^2+1),\ (a_1+b_1x) \div (x^2+1)^2.$$

Hence, the given fraction can be written in the form

$$\frac{x^4+4x^3+7}{(x-1)(x^2+1)^2} = \frac{A}{x-1} + \frac{a_0+b_0x}{x^2+1} + \frac{a_1+b_1x}{(x^2+1)^2} \quad . \qquad (1)$$

Multiplying (1) by $(x-1)(x^2+1)^2$, we have the identity

$$x^4+4x^3+7 = A(x^2+1)^2 + (a_0+b_0x)(x-1)(x^2+1) + (a_1+b_1x)(x-1). \qquad (2)$$

Putting $x=1$ in (2), we have $12=4A$ or $A=3$. (3)

Putting $x=i$ in (2), we get

$$1-4i+7 = (a_1+b_1i)(i-1) = -a_1-b_1+i(a_1-b_1).$$

Equating real and imaginary parts, we get

$$8=-a_1-b_1;\ -4=a_1-b_1;\ \text{hence } b_1=-2 \text{ and } a_1=-6. \qquad (4)$$

Making these substitutions in (2), we have

$$x^4+4x^3+7 = 3(x^2+1)^2 + (a_0+b_0x)(x-1)(x^2+1) + (-6-2x)(x-1)$$

or

$$-2x^4+4x^3-4x^2+4x-2 = (a_0+b_0x)(x-1)(x^2+1). \qquad (5)$$

(5) being an identity, we expect that its left-hand member is divisible by $(x-1)(x^2+1)$; and we easily realise that it is the case. Carrying out the division, we have $-2x+2=a_0+b_0x$, which is an identity and hence we have $a_0=2$ and $b_0=-2$. (6)

a_0 and b_0 may also be determined thus:–

Equating coefficients of x^4 on both sides of (2), we have $1=A+b_0$. Remembering that $A=3$, we have $b_0=-2$. (7)

In the same way, equating constant terms on both sides of (2), we have $7=A-a_0-a_1$. Substituting the values of A and a_1, we have $a_0=2$.

8·4. We now give some **trigonometrical applications** of the foregoing theory.

Prove that, if $2na=\pi$

$$\frac{1}{x^{2n}+1} = \frac{1}{n} \sum_{r=1}^{r=n} \frac{1-x\cos(2r-1)a}{1-2x\cos(2r-1)a+x^2} \ .$$

[c. f. *B. U. 1921 Hons. II*]

We know from 186 § 7·6, that $x-w$ where

$$w = \cos(2r-1)a + i\sin(2r-1)a$$

is a factor of $x^{2n}+1=0$. The partial fraction corresponding to this is $A/(x-w)$ where A is obtained by 202 putting w for x in

$$1 \div \frac{d}{dx}(x^{2n}+1) \text{ or } 1 \div 2nx^{2n-1},$$

and hence it is equal to $-w \div 2n$. Similarly the partial fraction corresponding to the conjugate factor $x-w^{-1}$ is

$$-w^{-1} \div 2n(x-w^{-1}).$$

The sum of these two fractions is

$$\frac{-1}{2n}\left\{\frac{w}{x-w}+\frac{w^{-1}}{x-w^{-1}}\right\} \text{ or } \frac{1}{n}\cdot\frac{1-x\cos(2r-1)a}{1-2x\cos(2r-1)a+x^2}.$$

Substituting $r=1, 2, 3, \ldots n$, we get n partial fractions whose sum is equal to the given fraction.

This result may also be obtained by differentiating with respect to x the identity of (§7·6),

$x^{2n}+1=(x^2-2x\cos a+1)(x^2-2x\cos 3a+1)(x^2-2x\cos 5a+1)\ldots n$ factors.

8·5. Prove that

cf. Hob 239 (4)
D & R 235·8

$$\frac{\cos mx}{\cos nx-\cos na} = \frac{1}{n\sin na}\sum_{r=0}^{r=n-1}\frac{\sin(a+rb).\cos m(a+rb)}{\cos x-\cos(a+rb)}$$

where m and n are positive integers $(m<n)$ and $2\pi = nb$.

We know from 94 (4) of §4·4, that $\cos mx$ and $\cos nx-\cos na$ can be expanded into polynomials in $\cos x$ of degrees m and n. Further $\cos nx-\cos na$ has 192 factors (7·2 Cor. 2)

$\cos x-\cos a$, $\cos x-\cos(a+b)$, $\cos x-\cos(a+2b)$, $\ldots \cos x-\cos[a+(n-1)b]$

Now the partial fraction, corresponding to one of the factors $\cos x-\cos(a+rb)$ is $A_r \div [\cos x-\cos(a+rb)]$ where A_r is obtained 202 by writing $\cos(a+rb)$ for $\cos x$ or $a+rb$ for x in

$$\cos mx \div \left[\frac{d}{d(\cos x)}(\cos nx-\cos na)\right] \text{ or } \frac{\sin x\cos mx}{n\sin nx}.$$

Hence $A_r = \sin(a+rb).\cos m(a+rb) \div (n\sin na)$.

By writing $r=1, 2, 3, \ldots, n$ in (1) and adding the n partial frations, we get the required result.

8·51. If $2na=\pi$, prove that

$$\frac{\sin 2x}{\sin 2nx}=\frac{1}{2n}\sum_{r=1}^{r=n-1}(-1)^{r-1}\frac{\sin^2 2ra}{\sin^2 ra-\sin^2 x}.$$

This result may be derived from the last article by suitable substitutions. An independent proof is however simple.

We know from (9) of §4·4, that $\sin 2nx \div \sin 2x$ can be expanded into a polynomial in $\sin^2 x$, and that its factors are

$$\sin^2 a-\sin^2 x,\ \sin^2 2a-\sin^2 x,\ \sin^2 3a-\sin^2 x,\ldots\sin^2(n-1)a-\sin^2 x.$$

The partial fraction corresponding to

$$\sin^2 ra-\sin^2 x \text{ is } A/(\sin^2 x-\sin^2 ra)$$

where A is obtained by putting $\sin^2 ra$ for $\sin^2 x$ or ra for x in

$$1\div\left[\frac{d}{dx(\sin^2 x)}\cdot\frac{\sin 2nx}{\sin 2x}\right] \text{ or } \frac{\sin^2 2x.\ 2\sin x.\cos x}{2n\sin 2x\cos 2nx-2\cos 2x\sin 2nx}.$$

Hence A is equal to $(-1)^r(\sin^2 2ra)/2n$. The other partial fractions are obtained by assigning to r, values 1, 2, 3, ...$n-1$.

8·52. If $4na=\pi$, prove that $2n\sec 2nx$ is equal to

$$\frac{\sin 2a}{\sin^2 a-\sin^2 x}-\frac{\sin 6a}{\sin^2 3a-\sin^2 x}+\frac{\sin 10a}{\sin^2 5a-\sin^2 x}-\ldots\text{to } n \text{ terms}.$$

The proof is left to the student as an exercise.

Examples. VIII. c.

Resolve the following into partial fractions:

1. $\dfrac{x}{(1+x)^2(1+x+x^2)}$.

2. $\dfrac{3x}{(1+x)(1+x^3)}$.

3. $\dfrac{1-4x+x^2}{(1-x)^3(1+x^2)}$.

4. $\dfrac{1+2x+3x^2}{(1+x)^3(1+x^2)}$.

5. $\dfrac{1-x}{(1+x)(1+x^2)^2}$.

6. $\dfrac{x^2+x+2}{(x^3+x-1)(x^2+x+1)}$.

7. Prove that, if $f(x)$ is a polynomial of lower degree than $n+1$

$$\frac{(-1)^n n!f(x)}{x(x-1)(x-2)\ldots(x-n)}=\frac{{}_nC_0 f(0)}{x}-\frac{{}_nC_1 f(1)}{x-1}+\frac{{}_nC_2 f(2)}{x-2}+\ldots$$
$$\ldots+\frac{(-1)^{n-1}{}_nC_{n-1}f(n-1)}{x-n+1}+\frac{(-1)^n{}_nC_n f(n)}{x-n}.$$

Prove the identities 8–11:

8. $$\frac{1}{1+x}+\frac{2x}{1+x^2}+\frac{4x^3}{1+x^4}+\frac{8x^7}{1+x^8}+\ldots\text{to } n \text{ terms} = \frac{px^{p-1}}{x^p-1}+\frac{1}{1-x}$$

where $p=2^n$.

9. $$\frac{3a^2b^2\sin 3x}{a^6-2a^3b^3\cos 3x+b^6} = \frac{\sin x}{a^2-2ab\cos x+b^2} + \frac{\sin(x+120^0)}{a^2-2ab\cos(x+120^0)+b^2} + \frac{\sin(x+240^0)}{a^2-2ab\cos(x+240^0)+b^2}.$$

10. $$\frac{5\sin 3x}{\sin 5x} = \frac{\sin^2 72^0}{\cos^2 x-\cos^2 36^0} + \frac{\sin^2 36^0}{\cos^2 x-\cos^2 72^0}.$$

11. $$\frac{7\cos 5x}{\cos 7x} = \frac{\sin^2 2a}{\cos^2 x-\cos^2 a} + \frac{\sin^2 6a}{\cos^2 x-\cos^2 3a} + \frac{\sin^2 10a}{\cos^2 x-\cos^2 5a}$$

where $14a=\pi$.

12. Prove that if $4na=\pi$

$$\frac{(1+x)^{2n}-(1-x)^{2n}}{(1+x)^{2n}+(1-x)^{2n}} = \frac{x}{n}\left\{\frac{\sec^2 a}{x^2+\tan^2 a}+\frac{\sec^2 3a}{x^2+\tan^2 3a}+\frac{\sec^2 5a}{x^2+\tan^2 5a}+\ldots+\frac{\sec^2(2n-3)a}{x^2+\tan^2(2n-3)a}+\frac{\sec^2(2n-1)a}{x^2+\tan^2(2n-1)a}\right\}.$$

Deduce by putting $x=i\tan y$ or prove otherwise that

$$\frac{n\tan 2ny}{\tan y} = \frac{\sec^2 a}{\tan^2 a-\tan^2 y}+\frac{\sec^2 3a}{\tan^2 3a-\tan^2 y}+\frac{\sec^2 5a}{\tan^2 5a-\tan^2 y}+\ldots.$$

13. Prove that if $4na=\pi$

$$\frac{2n}{(1+x)^{2n}+(1-x)^{2n}} = \frac{\sin a.\cos^{2n-3}a}{x^2+\tan^2 a}-\frac{\sin 3a.\cos^{2n-3}3a}{x^2+\tan^2 3a}+\frac{\sin 5a.\cos^{2n-3}5a}{x^2+\tan^2 5a}-\frac{\sin 7a.\cos^{2n-3}7a}{x^2+\tan^2 7a}+\ldots\text{to } n \text{ termas}$$

Deduce or otherwise prove that

$$\frac{n\cos^{2n}x}{\cos 2nx} = \frac{\sin a.\cos^{2n-3}a}{\tan^2 a-\tan^2 x}-\frac{\sin 3a.\cos^{2n-3}3a}{\tan^2 3a-\tan^2 x}+\ldots\text{to } n \text{ terms.}$$

14. Prove that

$$\frac{n(x^{2n-1}-a^n x^{n-1}\cos np)}{x^{2n}-2a^n x^n\cos np+a^{2n}} = \sum_{r=0}^{r=n-1}\frac{x-a\cos(p+2r\pi/n)}{x^2-2ax\cos(p+2r\pi/n)+a^2}.$$

[Take logarithms of the identity of § 7·7, and then differentiate with respect to x.]

15. Prove that

$$\sum_{r=0}^{r=n-1} \frac{1}{x^2-2ax\cos(p+2r\pi/n)+a^2} = \frac{n(x^{2n}-a^{2n})}{(x^2-a^2)(x^{2n}-2a^nx^n\cos np+a^{2n})}.$$

[Multiply the identity of example 14, above, by x; subtract n from the left hand fraction and 1 from each of the n right hand fractions]

Interpret the result geometrically.

16. Shew that if n is a positive integer and $2na=\pi$

$$2n\sec nx = \sum \sin ra \operatorname{cosec}\tfrac{1}{2}(ra-x)\operatorname{cosec}\tfrac{1}{2}(ra+x)$$

where r has the values 1, 5, 9,..., $4n-3$.

[*B. U. 1923 Hons* II.]

17. Prove that, if $na=\pi$,

I. $$\frac{n(\sin nx-\sin ny)}{\cos nx-\cos ny} = \sum_{0}^{n-1} \frac{\sin x-\sin(y+2ra)}{\cos x-\cos(y+2ra)};$$

II. $$n\cot ny = \cot y+\cot(y+a)+\cot(y+2a)+\ldots\text{to } n \text{ terms.}$$

[*B. U. 1927 Hons.* II.]

e.g. 219

CHAPTER IX

SUMMATION OF FINITE SERIES

9·0. The student has already studied the Arithmetical and the Geometrical series. He also knows how to find the sum of the squares and cubes of the first n natural numbers. We propose to study some more standard types of series and indicate some general methods of obtaining their sums in this chapter. We shall generally denote the rth term and the sum to r terms, by T_r and S_r respectively.

9·1. The sum of a series can easily be found if the rth term T_r, can be thrown into the form $k[t_r - t_{r-1}]$ where t_{r-1} is the same function of t_{r-1} as t_r is of r, k being independent of r.

For, putting $r = 2, 3, 4, \ldots n$, successively in the relation,

$$T_r = k\,(t_r - t_{r-1}),$$

we have

$$T_2 = k(t_2 - t_1),$$

$$T_3 = k(t_3 - t_2),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

and

$$T_n = k\,(t_n - t_{n-1}).$$

Adding up these relations, we have

$$S_n = T_1 + k\,(t_n - t_1).$$

No general method can be given for carrying out the transformation of T_r in the required form. The necessary skill must be acquired by solving a large number of examples.

Example 1. Sum the series to n terms

$$\frac{1}{4.1^2-1} + \frac{1}{4.2^2-1} + \frac{1}{4.3^2-1} + \frac{1}{4.4^2-1} + \ldots\ldots$$

Here

$$T_r = \frac{1}{4.\,r^2-1} = \frac{1}{(2r-1)(2r+1)}.$$

Now applying the rule for partial fractions, we find that

$$T_r = \tfrac{1}{2}\left\{\frac{1}{2r-1} - \frac{1}{2r+1}\right\}.$$

Putting $r=1, 2,\ldots n$ in this relation successively, we have

$$T_1+T_2+\ldots+T_n=\tfrac{1}{2}\left[\left(\tfrac{1}{1}-\tfrac{1}{3}\right)+\left(\tfrac{1}{3}-\tfrac{1}{5}\right)+\left(\tfrac{1}{5}-\tfrac{1}{7}\right)+\ldots\text{to } n \text{ term}\right]$$

$$=\tfrac{1}{2}\left\{\frac{1}{1}-\frac{1}{2n+1}\right\}=\frac{n}{2n+1}.$$

Example 2. Prove that, to n terms

$$\frac{5}{1.3.7.9}+\frac{7}{3.5.9.11}+\frac{9}{5.7.11.13}+\ldots=\frac{1}{28}-\frac{1}{4(2n+1)(2n+7)}.$$

It is easily seen that

$$T_r=\frac{2r+3}{(2r-1)(2r+1)(2r+5)(2r+7)}.$$

$$=\tfrac{1}{4}\frac{[(2r+1)(2r+7)-(2r-1)(2r+5)]}{(2r-1)(2r+1)(2r+5)(2r+7)},$$

$$\therefore \quad T_r=-\tfrac{1}{4}\left\{\frac{1}{(2r+1)(2r+7)}-\frac{1}{(2r-1)(2r+5)}\right\}.$$

Putting $r=1, 2, 3,\ldots n$ successively in this relation, we get

$$T_1+T_2+T_2+\ldots+T_n$$

$$=-\tfrac{1}{4}\left\{\left(\frac{1}{3.9}-\frac{1}{1.7}\right)+\left(\frac{1}{5.11}-\frac{1}{3.9}\right)+\left(\frac{1}{7.13}-\frac{1}{5.11}\right)+\ldots\text{to terms}\right\}$$

$$=-\tfrac{1}{4}\left\{\frac{1}{(2n+1)(2n+7)}-\frac{1}{1.7}\right\}=\frac{1}{28}-\frac{1}{4(2n+1)(2n+7)}.$$

Note:–If the sum is given as in this example the necessary transformation of T_r is suggested by the obvious fact that

$$T_r=S_r-S_{r-1}.$$

Example 3. Sum to n terms the series

$$(\cos^2x.\cot x-\sin^2x.\tan x)^{-1}+(\cos^23x.\cot x-\sin^23x.\tan x)^{-1}$$
$$+(\cos^25x\cot x-\sin^25x.\tan x)^{-1}\ldots$$
$$+[\cos^2(2n-1)x\cot x-\sin^2(2n-1)x\tan x]^{-1}$$

[*B. U. 1917, Hons.* II.]

It is easy prove that T_r reduces to

$$\frac{\sin 2x}{2\cos 2rx.\ \cos(2r-2)x} \quad \text{or} \quad \frac{\sin[2r-(2r-2)]x}{2\cos 2rx.\ \cos(2r-2)x},$$

$$\text{or} \quad \frac{\sin 2rx\cos(2r-2)x-\cos 2rx\sin(2r-2)x}{2\cos 2rx.\ \cos(2r-2)x},$$

$$\text{or} \quad \tfrac{1}{2}\left[\tan 2rx-\tan 2(r-1)x\right].$$

Hence, putting $r=1, 2, \ldots n$ in this relation, we get

$S_n=\frac{1}{2}[(\tan 2x-0)+(\tan 4x-\tan 2x)+(\tan 6x-\tan 4x)+\ldots$ to n terms$]$

$=\frac{1}{2}\tan 2nx.$

Example 4. Sum the series to n terms

$$\frac{5}{1.2.3}+\frac{6}{2.3.4}\cdot\frac{1}{2}+\frac{7}{3.4.5}\cdot\frac{1}{2^2}+\frac{8}{4.5.6}\cdot\frac{1}{2^3}+\ldots .$$

Here $$T_r = \frac{r+4}{r(r+1)(r+2)}\cdot\frac{1}{2^{r-1}}$$

$$= \frac{1}{2^{r-2}}\cdot\left\{\frac{1}{r(r+1)}-\frac{1}{2}\,\frac{1}{(r+1)(r+2)}\right\}.$$

Hence, putting $r=1, 2, 3\ldots n$ in this relation, it is easily seen that

$$S_n = 1-\frac{1}{2^{n-1}}\cdot\frac{1}{(n+1)(n+2)}.$$

Examples. IX. a.

Sum the following series to n terms:

1. $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\ldots .$

2. $\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+\ldots .$

3. $\frac{1}{1.5}+\frac{1}{3.7}+\frac{1}{5.9}+\frac{1}{7.11}+\ldots .$

4 $1!.1+2!.2+3!.3+4!.4+\ldots$

5. $\frac{1}{1\,3}-\frac{1}{2.4}+\frac{1}{3.5}-\frac{1}{4.6}+\ldots .$

6. $\frac{5}{4.1^2-1}+\frac{7}{4.2^2-1}\cdot\frac{1}{2}+\frac{9}{4.3^2-1}\cdot\frac{1}{2^2}+\ldots$

7. $\frac{1}{1.5}-\frac{2.3}{3.7}+\frac{3.3^2}{5.9}-\frac{4.3^3}{7.11}+\ldots .$

8. $\frac{1}{1+1^2+1^4}+\frac{2}{1+2^2+2^4}+\frac{3}{1+3^2+3^4}+\ldots$

9. $2\tan 2y+2^2\tan 2^2y+2^3\tan 2^3y+\ldots$

10. $\frac{1}{2}\tan\frac{1}{2}x+\frac{1}{4}\tan\frac{1}{4}x+\frac{1}{8}\tan\frac{1}{8}x+\dots$

11. $\sin x.\sin x+\sin 2x.\sin 2^2x+\sin 3x.\sin 3^2x+\dots$

12. $\tan^{-1}\frac{2}{2^2}+\tan^{-1}\frac{2}{4^2}+\tan^{-1}\frac{2}{6^2}+\tan^{-1}\frac{2}{8^2}+\dots$.

13. $\sin^3x+3\sin^3\frac{1}{3}x+9\sin^3\frac{1}{9}x+27\sin^3\frac{1}{27}x+\dots$.

14. $\frac{\sin x}{\cos x+\cos 2x}+\frac{\sin 2x}{\cos x+\cos 4x}+\frac{\sin 3x}{\cos x+\cos 6x}+\dots$.

9·2. *Find the sum of n terms of the series whose rth term is*

$$[a+rb][a+(r+1)b][a+(r+2)b][a+(r+3)b]\dots[a+(r+s-1)b]$$

The form of the series possesses the following characteristics (1) each term is a product of the same number of factors; (2) these factors are in *A. P.* and (3) the first factors of all these terms are also in *A. P.* the common difference being the same as in the former *A. P.*

Denote by t_r, $T_r\,[a+(r+s)b]$ or

$$[a+rb][a+(r+1)b][a+(r+2)b]\dots[a+(r+s)b]$$

Then we easily see that

$$t_r-t_{r-1}=T_r\,[a+(r+s)b-(a+\overline{r-1}b)]=T_r(s+1)b.$$

Hence, putting $r=1,\ 2,\ 3,\dots,\ n$ in the formula we have

$$t_1-t_0=T_1(s+1)b,$$
$$t_2-t_1=T_2(s+1)b,$$
$$\dots\dots\dots\dots\dots$$
$$t_n-t_{n-1}=T_n(s+1)b.$$

Adding up these relations, we have

$$t_n-t_0=(s+1)\ b\ S_n$$

or
$$S_n=\frac{t_n-t_0}{b(s+1)}.$$

Alternative methods will be suggested later on.

Cor. The student should notice the following important particular case:—

$$1.2.3.\dots s+2.3.4.5\dots(s+1)+\dots\dots+n(n+1)(n+2)\dots(n+s-1)$$
$$=[n(n+1)(n+2)(n+3)\dots(n+s)]\div(s+1)$$

It is obtained by putting $a=0,\ b=1$ in the above result.

Example 1. Sum the series to n terms

$$1.3.5+3.5.7+5.7.9+\ldots$$

Here $$T_r=(2r-1)(2r+1)(2r+3)$$

and $$t_r=(2r-1)(2r+1)(2r+3)(2r+5).$$

$$t_r-t_{r-1}=(2r-1)(2r+1)(2r+3)(2r+5)-(2r-3)(2r-1)(2r+1)(2r+3)$$
$$=8(2r-1)(2r+1)(2r+3).$$
$$\therefore\ t_r-t_{r-1}=8\,T_r.$$

Hence, putting $r=1, 2,\ldots n$ successively in this formula, and adding the n equations, we have

$$8\,S_n=(t_n-t_0)=(2n-1)(2n+1)(2n+3)(2n+5)+1.3.5.$$

Example 2. Sum the series to n terms

$$1.2.3^2+2.3.4^2+3.4.5^2+\ldots\ldots$$

Here $$T_r=\frac{r(r+1)(r+2)^2}{}$$
$$=r(r+1)(r+2)(r+3-1)$$
$$=r(r+1)(r+2)(r+3)-r(r+1)(r+2).$$

Hence, putting $r=1, 2,\ldots n$, we see that

$$S_n=(1.2.3.4+2.3.4.5+3.4.5.6+\ldots\text{to } n \text{ terms})$$
$$-(1.2.3+2.3.4+3.4.5+\ldots\text{to } n \text{ terms})$$

Now by § 9·2, Cor. 1, we have

$$1.2.3.4+2.3.4.5+3.4.5.6+\ldots\text{to } n \text{ terms}$$
$$=\tfrac{1}{5}\,n(n+1)(n+2)(n+3)(n+4);$$

and $$1.2.3+2.3.4+3.4.5+\ldots\text{to } n \text{ terms}$$
$$=\tfrac{1}{4}\,n(n+1)(n+2)(n+3)$$

$$\therefore\ S_n=\tfrac{1}{5}\,n(n+1)(n+2)(n+3)(n+4)-\tfrac{1}{4}n(n+1)(n+2)(n+3).$$
$$=\tfrac{1}{20}\,n(n+1)(n+2)(n+3)(4n+11).$$

9·201. Note how in the above example, the rth term is transformed and the series is thus split up into a number of series each resembling, in form, the series of §9·2, cor. 1. Generally, if T_r is a polynomial in r of the sth degree, then we can put it into the form, (Read remarks on illustrative example 1, § 1·33)

$$A_0+A_1r+A_2r(r+1)+A_3r(r+1)(r+2)+\ldots$$
$$+A_sr(r+1)(r+2)\ldots(r+s-1).$$

Thus putting $r=1, 2, 3, \ldots n$, we see that the series can be split up into $s+1$ series:

$$A_0(1+1+1+\ldots \text{to } n \text{ terms}),$$
$$A_1(1+2+3+\ldots \text{to } n \text{ terms}),$$
$$A_2(1\cdot2+2\cdot3+\ldots \text{to } n \text{ terms}),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$A_s[1.2\,3\ldots s+2.3.4.5\ldots(s+1)+\ldots \text{to } n \text{ terms}].$$

Hence, by §9·2, cor. 1, the sum of the series

$$= A_0 n + \tfrac{1}{2}A_1 n(n+1) + \tfrac{1}{3}A_2 n(n+1)(n+2) + \ldots$$

Example 1. Sum the series to n terms

$$1.1.1+2.3.4+3.5.7+4.7.10+\ldots$$

Here $T_r = r(2r-1)(3r-2)$.

$$\therefore \quad T_r = 6r(r+1)(r+2) - 25.r(r+1) + 15r$$

Hence the sum of the series to n terms, by §9·2. cor. 1,

$$= \frac{6}{4}n(n+1)(n+2)(n+3) - \frac{25}{3}n(n+1)(n+2) + \frac{15}{2}n(n+1)$$
$$= n(n+1)(9n^2-5n-1)/6.$$

Examples. IX. b.

Sum to n terms the following series:

1. $1.2^2+2.3^2+3.4^2+4.5^2+\ldots$.
2. $1.1^2+2.3^2+3.5^2+4.7^2+\ldots$.
3. $1^2.2^2+2^2.3^2+3^2.4^2+\ldots$.
4. $1.3.4+2.5.6+3.7.8+\ldots$.
5. $1.(2n-1)+3.(2n-3)+5.(2n-5)+\ldots$.
6. $(a+n)(b+1)+(a+n-1)(b+2)+(a+n-2)(b+3)+\ldots$
7. $n.1.2+(n-1)2.3+(n-2)3.4+(n-3)4.5+\ldots$
8. $1.2+(1.2+2.3)+(1.2+2.3+3.4)+(1.2+2.3+3.4+4.5)+\ldots$
9. $1.1(3n-2)+2.3(3n-5)+3.5.(3n-8)+\ldots$.

9·21. We denote *the sum of the pth powers of the first n natural numbers* by ${}_pS_n$. In this case $T_r = r^p$, a polynomial in r and hence the method of §9·21, can be successfully applied to it. Thus

$${}_1S_n = 1+2+3+\ldots+n = \tfrac{1}{2}n(n+1)$$
$${}_2S_n = 1^2+2^2+3^2+\ldots\ldots n^2 = \Sigma[r(r+1)-r]$$
$$= \tfrac{1}{3}n(n+1)(n+2) - \tfrac{1}{2}n(n+1)$$
$$= \tfrac{1}{6}n(n+1)(2n+1).$$

In the same way, it is easy to see that

$$_3S_n = \tfrac{1}{4}n^2(n+1)^2 \text{ or } (_1S_n)^2.$$

Another method of obtaining such results directly will be suggested in the chapter on exponential series.

9·211. The following formula connects $_pS_n$ with the $p-1$ sums $_{p-1}S_n$, $_{p-2}S_n$, $_{p-3}S_n$,...$_1S_n$.

Prove the formula

$$(n+1)^{p+1}-(n+1) = {}_{p+1}C_1 \cdot {}_pS_n + {}_{p+1}C_2 \cdot {}_{p-1}S_n + \ldots + {}_{p+1}C_{p1} \cdot {}_1S_n.$$

We have, by the binomial theorem for a positive integral exponent,

$$(r+1)^{p+1}-r^{p+1} = {}_{p+1}C_1 r^p + {}_{p+1}C_2 r^{p-1} + {}_{p+1}C_3 r^{p-2} + \ldots {}_{p+1}C_p r + 1.$$

Putting $r = 1, 2, 3, \ldots n$, in this relation and adding up the n equations thus obtained, we have

$$(n+1)^{p+1}-(n+1) = {}_{p+1}C_1 \cdot {}_pS_n + {}_{p+1}C_2 \cdot {}_{p-1}S_n + \ldots + {}_{n+1}C_p \cdot {}_1S_n.$$

Example 1. If $S_m = 1^m + 2^m + 3^m + \ldots + n^m$ prove that

$$(1) \quad 3S_5 + S_3 = 4(S_1)^3$$

and

$$(2) \quad S_5 + S_7 = 2(S_3)^2.$$

[*B. U., 1914, Problems.*]

We observe that

$$r^3(r+1)^3-(r-1)^3r^3 = 6r^5 + 2r^3.$$

Putting $r = 1, 2, 3, \ldots n$ in this identity and adding up the resulting relations, we have

$$n^3(n+1)^3 = 6S_5 + 2S_3.$$

Remembering that $S_1 = \frac{1}{2}n(n+1)$, we have at once

$$8(S_1)^3 = 6S_5 + 2S_3$$

from which the first part is obtained.

To obtain the second part, we observe that

$$r^4(r+1)^4-(r-1)^4r^4 = 8r^7 + 8r^5.$$

Putting $r = 1, 2, 3 \ldots n$ in this relation and adding up the results, as before, we easily have

$$n^4(n+1)^4 = 8(S_7 + S_5).$$

Now substitute $16\,(S_3)^2$ for $n^4(n+1)^4$.

Example 2. Show that $12\,S_5 = (n+1)(2n^3+4n^2+n-1)$.
[*B. U. 1922, Hons.*, I.]

By example 1 above, we have

$$3S_5 = 4(S_1)^3 - 1S_3$$

Now $S_1 = \frac{1}{2}n(n+1)$ and $S_3 = \frac{1}{4}n^2(n+1)^2$ and hence

$$12S_5 = 2n^3(n+1)^3 - n^2(n+1)^2 = n^2(n+1)^2(2n^2+2n-1).$$

Examples. IX. c.

1. From the identity

$$n^2(n+1)^2(2n+1) - (n-1)^2n^2(2n-1) = 10n^4 + 2n^2,$$

deduce that (1) $n^2(n+1)^2(2n+1) = 10_4S_n + 2_2S_n$

and that (2) $_4S_n = \frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1)$.

2. Prove that

$$_7S_n = \frac{1}{24}n^2(n+1)^2(3n^4+6n^3-n^2-4n+2).$$

3. Prove that the sum of the products of the first n natural numbers taken two at a time is $\frac{1}{24}n(n+1)(n-1)(3n+2)$.

4. Prove by using the identity

$$(2x+1)^{p+1} - (2x-1)^{p+1} = 2({}_{p+1}C_1 2^p x^p + {}_{p+1}C_3 2^{p-2}x^{p-2} + \ldots)$$

that $(2n+1)^{p+1} - 1$

$$= {}_{p+1}C_1 2^{p+1}\,{}_pS_n + {}_{p+1}C_3 2^{p-1}\,{}_{p-2}S_n + {}_{p+1}C_5 2^{p-3}\,{}_{p-4}S_n + \ldots$$

Prove further that the last term in the right hand expression of the latter result is either $2n$ or $4\,{}_{p+1}C_p\,S_n$ according as p is even or odd.

9·22. Consider the following series of numbers:

1,	1,	1,	1,	1,	1,	.....
1,	2,	3,	4,	5,	6,	.....
1,	3,	6,	10,	15,	21,	.....
1,	4,	10,	20,	35,	56,	.....
1,	5,	15,	35,	70,	126,	.....

These series possess the following characteristics:

(1) Every term of the first series is equal to one. (2) The first term of every series is also equal to one. (3) The r th term of a series is equal to the sum of the first r terms of the preceding

series. These numbers are called **figurate numbers**; the terms of the rth series are said to be of the rth order and we shall see later on that they are coefficients in the expansion of $(1-x)^{-r}$.

Denoting the $s+1$ th term of the $r+1$ th series by ${}_ra_s$, we have immediately from cor. 1, § 9.2,

$${}_1a_{n-1} = 1+1+1+\ldots \text{to } n \text{ terms} = n;$$
$${}_2a_{n-1} = 1+2+3+\ldots+n = \tfrac{1}{2}n(n+1).$$
$${}_3a_{n-1} = 1+3+6+\ldots+\tfrac{1}{2}n(n+1) = \tfrac{1}{3}n(n+1)(n+2)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

In this way we find generally that

$${}_ra_{n-1} = n(n+1)(n+2)\ldots(n+r-1) \div r = \frac{(n+r-1)!}{r!\,(n-1)!}$$

Examples. IX. d.

Prove the following relations between figurate numbers:

1. ${}_ra_n = {}_ra_{n-1} + {}_{r-1}a_n$.
2. ${}_1a_n = {}_ra_{n-1} + {}_{r-1}a_{n-1} + {}_{r-2}a_{n-1} + \ldots + {}_{r-p}a_{n-1} + {}_{r-p-1}a_n$.
3. ${}_ra_n = {}_ra_p + {}_{r-1}a_{p+1} + {}_{r-1}a_{p+2} + \ldots + {}_{r-1}a_n$.
4. ${}_ma_n = {}_na_m$.
5. ${}_ra_0, {}_{r-1}a_1, {}_{r-2}a_2, \ldots {}_1a_{r-1}, {}_0a_r$ are the successive coefficients in the expansion of $(1+x)^r$.

6. Prove that the first $n+1$ coefficients in the expansions of $P(x)$, $[P(x)]^2$. $[P(x)]^3$... are the first $n+1$ terms of the figurate numbers of the 1st, 2nd, 3rd...orders respectively, $P(x)$ being the polynomial $1+x+x^2+\ldots+x^n$.

9·23. The sum of the first r terms of the A- P. whose first term is 1 and common difference, d, is $\frac{1}{2}r[2+d(r-1)]$. Setting $d=0, 1, 2,\ldots$successively, we have the series of numbers

$$r,\ \tfrac{1}{2}r(r+1),\ r^2,\ \tfrac{1}{2}r(3r-1),\ \tfrac{1}{2}r(4r-2),\ldots\ldots$$

The following set of series is formed by choosing these numbers as their rth terms:

0 ... 1, 2, 3, 4, ... r.....;

1 ... 1, 3, 6, 10, ... $\frac{1}{2}r(r+1)$...:

2 ... 1, 4, 9, 16, ... r^2 ...;

.....

r ... $\frac{1}{2}r(3r+3-4)$

r ... $\frac{1}{2}r(4r+4-6)$

r ... $\frac{1}{2}r(d\,r+4-2d)$

The numbers in the mth row or series are called **polygonal numbers** of the $m+1$th order, those of the first order being each equal to one. Polygonal numbers of the second, third, fourth,... orders, are simetimes called linear, triangular, square... .

To find the sum of the first n terms of the mth order of polygonal numbers, we observe that the rth term in this case is $\frac{1}{2}r[2+(m-2)(r-1)]$ and hence, by cor. 1, § 9·2, the sum of the first n terms is

$$\tfrac{1}{2}n(n+1)+(m-2)n(n-1)(n+1)\div 6.$$

or
$$n(n+1)[3+(m-2)(n-1)]\div 6.$$

Example.

If $_m p_n$ denotes the nth term of the $m+1$ th order of polygonal numbers, prove that

$$1p_1,\ 2p_1,\ 3p_1,\cdots .$$

$$1p_2,\ 2p_2,\ 3p_2,\cdots .$$

$$1p_3,\ 2p_3,\ 3p_3,\cdots .$$

are arithmetical progressions whose first terms are 1, 2, 3, 4,... and whose common differences are 0, 1, 3, 6...*i. e.* 0 and the polygonal numbers of the third order.

9·3. *Sum the series to n terms whose r th term T_r is the reciprocal of*

$$[a+rb]\ [a+(r+1)b]\ [a+(r+2)b]\ldots[a+(r+s-1)b].$$

It should be noted that the reciprocals of the terms of this series form a series possessing the same characteristics as that of § 9·2. Now consider another series whose rth term t_r is derived from T_r by omitting the first factor in the denominator. We thus have
$$t_r=(a+rb)\ T_r$$
and so
$$t_r-t_{r-1}=b(1-s)\,T_r.$$

Putting $r=1,2,3,\ldots n$ for r in this relation and adding up the resulting n equations we have.

$$t_n-t_0=b(1-s)S_n,\text{ or }S_n=(t_n-t_0)\div b(1-s).$$

The theorem may also be established by integrating, between zero and one, both sides of the identity

$$(1-x)^s(x^m+x^{m+1}\ldots+x^{m+n-1})=(1-x)^{s-1}(x^m-x^{m+n})$$

and then transforming the result by suitable substitutions.

Example 1. sum the series to n terms

$$\frac{1}{1.4.7}+\frac{1}{4.7.10}+\frac{1}{7.10.13}+\ldots$$ [*B. U., 1924, Hons.*, I.]

It is obvious that the series coincides in form with that of § 9·3, and that in this case T_r and t_r are respectively equal to

$$\frac{1}{(3r-2)(3r+1)(3r+4)}, \quad \frac{1}{(3r+1)(3r+4)}.$$

Hence $t_r - t_{r-1} = \dfrac{1}{(3r+1)(3r+4)} - \dfrac{1}{(3r-2)(3r+1)}$

$$= \frac{-6}{(3r-2)(3r+1)(3r+4)} = -6\,T_r.$$

Writing 1, 2 3,...n for r in this relation and adding up the results thus obtained, we have

$$t_p - t_n = 6\,S_n$$

or $$S_n = \frac{1}{24} - \frac{1}{6}\,\frac{1}{(3n+1)(3n+4)}.$$

Example 2. If S_n denotes the sum to n terms of the series $1.2+2.3+3.4+\ldots$ and S'_{n-1} that to $n-1$ terms of the series

$$\frac{1}{1.2.3.4}+\frac{1}{2.3.4.5}+\ldots$$

shew that $18\,S_n.\,S'_{n-1} - S_n + 2 = 0$.

[*B. U. 1918, Hons.* I.]

By cor. 1 of § 9·2, we have

$$S_n = \tfrac{1}{3}n(n+1)(n+2)$$

and by § 9·3, we have

$$S'_{n-1} = \tfrac{1}{3}\left[\frac{1}{1.2.3} - \frac{1}{n(n+1)(n+2)}\right]$$

$$\therefore \qquad S'_{n-1} = \frac{1}{18} - \frac{1}{9}\cdot\frac{1}{S_n}$$

from which the desired result follows immediately.

Example 3. Find the sum to n terms of

$$\frac{1}{2.8.11}+\frac{1}{5.11.14}+\frac{1}{8.14.17}+\ldots.$$

[*B. U. 1916 Hons.* I]

In this case T_r is equal to

$$\frac{1}{(3r-1)(3r+5)(3r+8)} \text{ or } \frac{3r+2}{(3r-1)(3r+2)(3r+5)(3r+8)}$$

Now writing $(3r-1)+3$ for $3r+2$, we have

$$T_r = \frac{1}{(3r+2)(3r+5)(3r+8)} + \frac{3}{(3r-1)(3r+2)(3r+5)(3r+8)}$$

Putting $r=1, 2, 3, \ldots n$ in this relation, we have two series each of which coincides in form with that of §9·3; and hence their sums are

$$\frac{1}{6}\left\{\frac{1}{5.8} - \frac{1}{(3n+8)(3n+5)}\right\},$$

$$\frac{3}{9}\left\{\frac{1}{2.5.8} - \frac{1}{(3n+2)(3n+5)(3n+8)}\right\}$$

$$\therefore \quad S_n = \frac{1}{120} - \frac{3n+4}{6\,(3n+2)(3n+5)(3n+8)}.$$

The student should mark how the rth term is transformed and how the series is thus split up into a number of series each possessing the characteristics of the series of §9·3. This transformation can be always applied whenever the numerator of the rth term consists of a polynomeal in r and the denominator can be made to coincide in form with those of terms of the series of §9·3 The following example illustrates a more general method **the method of partial fractions.**

Example 4. Sum the series to n terms whose rth term is

$$\frac{8n}{(4n^2-1)^2}.$$

Putting T_r into partial fractions, we have

$$T_r = \frac{1}{(2r-1)^2} - \frac{1}{(2r+1)^2}.$$

$$\therefore \; S_n = \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{5^2} - \frac{1}{7^2} + \ldots + \frac{1}{(2n-1)^2} - \frac{1}{(2n+1)^2}$$

$$= 1 - \frac{1}{(2n+1)^2}.$$

Examples. IX. e.

Sum, to n terms, the series whose rth terms are

1. $\frac{1}{r(r+2)}$. 2. $\frac{1}{(2r-1)(2r+3)}$. 3. $\frac{1}{(3r-2)(3r+7)}$.

4. $\frac{1}{(3r-2)(3r+1)(3r+7)}$ 5. $\frac{1}{[a+(r-1)b][a+(r+2)b]}$.

Sum the following series to n terms:

6. $\frac{1}{1.2.3.} + \frac{3}{2.3.4} + \frac{5}{3.4.5} + \ldots$

[*B. U., 1926, Hons.* I.]

7. $\frac{1}{1^2} + \frac{3}{1^2+2^2} + \frac{5}{1^2+2^2+3^2} + \frac{7}{1^2+2^2+3^2+4^2} + \ldots$.

8. $\frac{n}{1.2\,3} + \frac{n-1}{2.3.4} + \frac{n-2}{3.4.5} + \ldots$.

9. $\frac{n(n+1)}{1.2.3.4} + \frac{(n-1)n}{2.3.4.5} + \frac{(n-2)(n-1)}{3.4.5.6} + \ldots$

10. $\frac{4}{1.2.2.5} + \frac{7}{2.3.5.8} + \frac{10}{3.4.8.11} + \ldots$.

9·4. *Sum the series to n terms,*

$$\frac{a}{b} + \frac{a(a+x)}{b(b+x)} + \frac{a(a+x)(a+2x)}{b(b+x)(b+2x)} + \ldots .$$

In this series T_r is given by

$$T_r = \frac{a[a+x][a+2x]\ldots[a+(r-1)x]}{b[b+x][b+2x]\ldots[b+(r-1)x]}.$$

We observe that the numerator of T_r consists of the product of r factors which are the first r terms of an $A.P.$ Similarly, the denominator also consists of the product of r factors which are the first r terms of another $A.P.$ the common difference in the two $A.P.$ s being the same.

$$\text{If } t_r = T_r\,(a+rx) = \frac{a(a+x)(a+2x)\ldots(a+rx)}{b(b+x)(b+2x)\ldots[b+(r-1)x]}$$

then it is easily seen that $t_{r-1} = [b+(r-1)x]\ T_r$;

so that $$t_r - t_{r-1} = T_r\,(a-b+x).$$

Putting $r=2, 3,\ldots n$ in this formula successively and adding up the $n-1$ results we have–

$$T_1(a-b+x)+S_n(a-b+x)\equiv t_n-t_1$$

or $$S_n = \frac{a}{a-b+x}\left\{\frac{(a+x)(a+2x)\ldots(a+nx)}{b(b+x)(b+2x)\ldots[b+(n-1)x]}-1\right\}.$$

The last result may also be obtained by integrating, between the limits zero and one, the following obvious identity

$$(1-x)^p(x^q+x^{q+1}+x^{q+2}+\ldots+x^{q+r-1})=(1-x)^{p-1}(x^q-x^{q+r})$$

and transforming the result by suitable substitutions. [See also[329] § 9·51 ~~cor. 1.~~ and *Examples*.. IX. k.[238], 13.]

9·41. Changing the sign of a in the series of § 9·4 and then transforming it, we have

$$\frac{a}{b}-\frac{a(a-x)}{b(b+x)}+\frac{a(a-x)(a-2x)}{b(b+x)(b+2x)}-\ldots \text{ to } n \text{ terms}$$

$$=\frac{a}{a+b-x}\left\{1-\frac{(-1)^n(a-x)(a-2x)(a-3x)\ldots(a-nx)}{b(b+x)(b+2x)\ldots[b+(n-1)x]}\right\}.$$

9·42. It is interesting to see how the series of §§ 9·2, 9·3, could be derived from that of § 9·4.

Thus the series $5.7+7.9+9.11+11.13+\ldots$ transforms into

$$5.7+\frac{5.7.9}{5}+\frac{5.7.9.11}{5.7}+\frac{5.7.9.11.13}{5.7.9}+\ldots.$$

$$\text{or } 1.3\left\{\frac{5.7}{1.3}+\frac{5.7.9}{1.3.5}+\frac{5.7.9.11}{1.3.5.7}+\ldots\right\}.$$

Similarly $$\frac{1}{5.7}+\frac{1}{7.9}+\frac{1}{9.11}+\frac{1}{11.13}+\ldots.$$

changes into

$$\frac{1}{5.7}+\frac{5}{5.7.9}+\frac{5.7}{5.7.9.11}+\frac{5.7.9}{5\,7.9.11.13}+\ldots$$

$$\text{or } \frac{1}{3}\left\{\frac{1.3}{5.7}+\frac{1.3.5}{5.7.9}+\frac{1.3.5.7}{5.7.9.11}+\frac{1.3.5.7.9}{5.7.9.11.13}+\ldots\right\}.$$

Example 1. Prove by any method that

$$1+\frac{x}{1!}+\frac{x(x+1)}{2!}+\frac{x(x+1)(x+2)}{3!}+\ldots+\frac{x(x+1)\ldots(x+n-1)}{n!}$$

$$=\frac{(x+1)(x+2)\ldots\ldots(x+n)}{n!}. \qquad [B.\ U.,\ 1921,\ Hons.,\ \text{I.}]$$

The series may be written in the form

$$1+\frac{x}{1}+\frac{x(x+1)(x+2)}{1(1+1)(1+2)}+\ldots\ldots .$$

We thus see that the series, after the first term, coincides in form with the series of § 9·4 and can be deduced from it by setting x, 1, 1 for a, b, x respectively in it. Hence

$$S_n = (x+1)(x+2)(x+3)\ldots(x+n) \div n!.$$

Example 2. show that the sum of the reciprocals of the binomial $(1-x)^{2n}$ is $2-1 \div (n+r)$.

[*B. U., 1921 Hons.*, I.]

We have to find the sum of the series

$$1-\frac{1}{2n}+\frac{1.2}{2n(2n-1)}-\frac{1.2.3}{2n(2n-1)(2n-2)}+\ldots \text{ to } 2n+1 \text{ terms}$$

$$\text{or } 1+\frac{1}{(-2n)}+\frac{1(1+1)}{(-2n)(-2n+1)}+\frac{1(1+1)(1+2)}{(-2n)(-2n+1)(-2n+2)}+\ldots$$

and hence its sum is

$$1+\frac{1}{2+2n}\left\{\frac{2.\ 3.\ 4.\ \ldots\ (1+2n)}{(-2n)(-2n+1)\ldots(-2n+2n-1)}-1\right\} \text{ or } 2-\frac{1}{n+1}.$$

The problem may also be solved by integrating, between zero and one, both sides of the identity

$$x^{2n+1}+(1-x)^{2n+1}=x^{2n}-x^{2n-1}(1-x)+x^{2n-2}(1-x)^2-\ldots+(1-x)^{2n}.$$

Example 3. shew that

$$\frac{{}_nC_1}{1-x}-\frac{{}_nC_2}{(1-x)(1-\frac{1}{2}x)}+\frac{{}_nC_3}{(1-x)(1-\frac{1}{2}x)(1-\frac{1}{3}x)}-\ldots=\frac{n}{n-x},$$

and hence or otherwise prove that

$${}_nC_1\sigma_1-{}_nC_2\sigma_2+{}_nC_3\sigma_3-\ldots=1\div n$$

where $\sigma_r=1+\frac{1}{2}+\frac{1}{3}+\ldots+1\div r.$

[*B. U., 1919, Hons., I.*]

Writing the given series in the form

$$-\frac{-n}{1-x}-\frac{(-n)(-n+1)}{(1-x)(1-x+1)}-\frac{(-n)(-n+1)(-n+2)}{(1-x)(1-x+1)(1-x+2)}-\ldots,$$

we see that it is identical in form with the series of § 9·4. Its sum is therefore equal to

$$-\frac{-n}{-n+1+x-1}\left\{\frac{(-n+1)(-n+2)\ldots(-n+n)}{(1-x)(1-x+1)\ldots(1-x+n-1)}-1\right\}=\frac{n}{n-x}.$$

To solve the other part of the problem, differentiate the identity with respect to x and obtain

$$\frac{{}_nC_1}{(1-x)^2} - \frac{{}_nC_2}{(1-x)(1-\frac{1}{2}x)}\left\{\frac{1}{1-x}+\frac{\frac{1}{2}}{1-\frac{1}{2}x}\right\}+\ldots=\frac{n}{(n-x)^2}.$$

Putting $x=0$ in this, the desired result is obtained at once.

The student who has studied Maclaurin's Theorem in the Calculus will see that the above method is equivalent to this: Expand both sides of the above identity in ascending powers of x and then equate the coefficients of x.

An independent solution is obtained by integrating, between zero and one, the identity of Examples. I. *b*., 19.

Examples. IX. f.

Sum the following series to n terms:

1. $\frac{2}{1}+\frac{2.5}{1.4}+\frac{2.5.8}{1.4.7}+\frac{2.5.8}{1.4.7}\frac{.11}{.10}+\ldots$.

2. $\frac{2.5.8}{7}+\frac{2.5.8.11}{7.10}+\frac{2.5\ 8.11.14}{7.10.13}+\ldots$.

3. $\frac{9}{3.7.11}+\frac{9.13}{3.7.11.15}+\frac{9.13.17}{3.7.11\ 15.19}+\ldots$.

4. $1-\frac{r-1}{r+1}+\frac{(r-1)(r-2)}{(r+1)(r+2)}-\frac{(r-1)(r-2)(r-3)}{(r+1)(r+2)(r+3)}+\ldots$.

5. $1-\frac{r+1}{r-1}+\frac{(r+1)(r+2)}{(r-1)(r-2)}-\frac{(r+1)(r+2)(r+3)}{(r-1)(r-2)(r-3)}+\ldots$.

6. Prove that the sum to n terms of the series

$$1+\frac{2}{2!}+\frac{1!2!2^3}{4!}+\frac{2!3!2^5}{6!}+\ldots=\frac{n!n!2^{2n}}{(2n)!}.$$

7. Prove that the sum to n terms of the series

$$\frac{1}{1}+\frac{2!}{1!1!2^2}+\frac{4!}{2!2!2^4}+\frac{6!}{3!3!2^6}+\ldots=\frac{(2n)!}{n!(n-1)!2^{2n-1}}.$$

9·5. Euler's Identity. The following obvious identity comprehends a large number, of series as particular cases

$$1.(1-a_1)+a_1(1-a_2)+a_1a_2(1-a_3)+\ldots+a_1a_2\ldots a_{n-1}(1-a_n)$$
$$=1-a_1a_2a_3\ldots a_n.$$

The proof is quite simple. We have

$$S_1 = T_1 = 1-a_1;$$
$$S_2 = S_1 + T_2 = 1-a_1 + a_1(1-a_2) = 1-a_1a_2;$$
$$S_3 = S_2 + T_3 = 1-a_1a_2 + a_1a_2(1-a_3) = 1-a_1a_2a_3;$$
$$S_4 = S_3 + T_4 = 1-a_1a_2a_3 + a_1a_2a_3(1-a_4) = 1-a_1a_2a_3a_4;$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

In this way it is easy to verify the identity.

9·51. The series of § 9·4 is most easily derived by simple substitutions in the above identity.

For, putting

$$a_1 = \frac{a+x}{b}\,,\ a_2 = \frac{a+2x}{b+x}\,,\ a_3 = \frac{a+3x}{b+2x}\,,\ \ldots\ .$$

we have

$$\frac{b-a-x}{b} + \frac{(a+x)(b-a-x)}{b(b+x)} + \frac{(a+x)(a+2x)(b-a-x)}{b(b+x)(b+2x)} + \ldots$$
$$= 1 - \frac{(a+x)(a+2x)(a+3x)\ldots(a+nx)}{b(b+x)(b+2x)\ldots[b+(n-1)x]}\,.$$

which reduces to the desired form when both sides are multiplied by $a \div (b-a-x)$.

Examples. IX. g.

Prove the following identities:

1. $(1+a_1)+(1+a_1)a_2+(1+a_1)(1+a_2)a_3+\ldots$ to n terms
$=(1+a_1)(1+a_2)(1+a_3)\ldots(1+a_n)$.

2. $\dfrac{x}{x+a_1} + \dfrac{a_1}{x+a_1}\cdot\dfrac{x}{x+a_2} + \dfrac{a_1}{x+a_1}\cdot\dfrac{a_2}{x+a_2}\cdot\dfrac{x}{x+a_3} + \ldots$

to n terms $= 1 - \dfrac{a_1a_2a_3\ldots a_n}{(x+a_1)(x+a_2)\ldots(x+a_n)}$.

3 $\dfrac{a_1}{1+a_1} + \left(\dfrac{a_2}{(1+a_1)(1+a_2)} + \dfrac{a_3}{(1+a_1)(1+a_2)(1+a_3)} + \ldots\right.$

$$\text{to } n \text{ terms} = 1 - \frac{1}{(1+a_1)(1+a_2)\ldots(1+a_n)}\,.$$

4. $[(1+a_1)+(1+a_1)a_2+(1+a_1)(1+a_2)a_3+\ldots$to n terms$]$
$\times[(1-a_1)-(1-a_1)a_2-(1-a_1)(1-a_2)a_3-\ldots$to n terms$]$
$=[(1-a_1^2)-(1-a_1^2)a_2^2-(1-a_1^2)(1-a_2^2)a_3^2-$to n terms$]$.

9·6. The method of mathematical induction has a very wide range of application in all branches of mathematics; it comes to our aid when all other methods apparently fail. The student should have a perfect grasp of the underlying principle and is therefore advised to study the following examples carefully.

Example 1. Prove that

$$\sum_1^n (r+2)^{(3)} = 1.2.3+2.3.4+3.4.5+\ldots\text{to } n \text{ terms} = \tfrac{1}{4}n(n+1)(n+2)(n+3).$$

Assume that the result is true for a particular value m of n; so that

$$S_m = \tfrac{1}{4}m(m+1)(m+2)(m+3)$$

$$\therefore\ S_{m+1} = S_m + T_{m+1} = \tfrac{1}{4}m(m+1)(m+2)(m+3) + (m+1)(m+2)(m+3) = \tfrac{1}{4}(m+1)(m+2)(m+3)(m+4).$$

We thus see that if the result is true for a particular value, m of n, it is true for the next greater value $m+1$ of n. Now the result is evidently true for the value 1 of n; hence it is true for the value 2, and therefore for the value, 3 and so on successively. The result is therefore true for all values of n.

Example 2. In the notation of § 9·21, prove that ${}_rS_n$ is a polynomial in n of degree $r+1$, divisible by $n(n+1)(2n+1)$ or $n^2(n+1)^2$ according as r is even or odd, r being greater than unity.

From *Examples*. IX. e., 4 , we have

$$(2n+1)^{1+r}-1 = {}_{r+1}C_1 2^{r+1}\,{}_rS_n + {}_{r+1}C_3 2^{r-1}\,{}_{r-2}S_n + {}_{r+1}C_5 2^{r-3}\,{}_{r-4}S_n + \ldots$$

in which the last term is either $2n$ or ${}_{r+1}C_r 2^2\,{}_1S_n$ according as r is even or odd.

To prove that ${}_rS_n$ is a polynomial in n of the $r+1$th degree, assume that the theorem is true for values $r=1, 2, 3, \ldots r-1$. Now in the above relation $(2n+1)^{r+1}-1$ is a polynomial in n of the $r+1$th degree and ${}_{r+1}C_3 . 2^{r-1} . {}_{r-2}S_n + \ldots$ is a polynomial of the $r-1$th degree at most. Hence ${}_rS_n$ must be a polynomial of the $r+1$th degree. Now ${}_1S_n = \frac{1}{2}n(n+1)$, a polynomial of the second degree, $\therefore$ the theorem is true for the value $r=1$; it must be therefore true for the value $r=2$. Again, since the theorem is true for $r=1, 2$, it must be true for $r=3$. In this way the proof may be completed by induction.

To prove that ${}_rS_n$ is divisible by $n(n+1)(2n+1)$ when r is even, we observe that in this case $(2n+1)^{r+1}-(2n+1)$ is equal to

$${}_{r+1}C_1 2^{r+1}\,{}_rS_n + {}_{r+1}C_3 2^{r-1}\,{}_{r-2}S_n + \ldots + {}_{r+1}C_{r-1}2^3\,{}_2S_n.$$

Now assume that ${}_{r-2}S_n$, ${}_{r-4}S_n$, ${}_{r-6}S_n \ldots {}_2S_n$ are divisible by $n(n+1)(2n+1)$; and observe that

$$(2n+1)^{r+1}-(2n+1) \text{ or } (2n+1)[(2n+1)^r-1]$$

is divisible by $n(n+1)(2n+1)$. It follows that under this supposition ${}_rS_n$ must be divisible by $n(n+1)(2n+1)$.

Now ${}_2S_n = \frac{1}{2}n(n+1)(2n+1)$ and hence, ${}_4S_n$ must be divisible by $n(n+1)(2n+1)$. Since the theorem is true for $r=2$, 4 it must be true for $r=6$ and so on. The third part of the problem may be similarly established.

Examples. IX. h.

1. Prove that $n(n+1)1.2+(n-1)n.2.3+(n-2)(n-1).3.4+\ldots$ to n terms is equal to $n(n+1)(n+2)(n+3)(n+4)/30$.

2. Prove that $\dfrac{2+2^3}{(1.2.3)^4}+\dfrac{3+3^3}{(2.3.4)^4}+\dfrac{4+4^3}{(3.4.5)^4}+\ldots$

to n terms is equal to $\frac{1}{8}\left\{\frac{1}{16}-\dfrac{1}{(n+1)^4(n+2)^4}\right\}$.

Prove the following identities 3–7:

3. $\dfrac{{}_nC_0}{1.2}-\dfrac{{}_nC_1}{2.3}+\dfrac{{}_nC_2}{3.4}-\dfrac{{}_nC_3}{4.5}+\ldots=\dfrac{1}{n+2}$.

4. $\dfrac{{}_nC_1}{1^2}-\dfrac{{}_nC_2}{2^2}+\dfrac{{}_nC_3}{3^2}-\ldots+\dfrac{(-1)^{n-1}\,{}_nC_n}{n^2}$

$$= X_1+\tfrac{1}{2}X_2+\tfrac{1}{3}X_3+\tfrac{1}{4}X_4\ldots+(X_n\div n).$$

where $X_r=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots+(1\div r)$.

5. $1-m\,\dfrac{(2n+1)}{2n+2}+\dfrac{m(m-1)}{1.2}\,\dfrac{(2n+1)(2n+3)}{(2n+2)(2n+4)}-\ldots$

$$=\frac{1.3.5\ldots(2m-1)}{(2n+2)(2n+4)\ldots(2n+2m)}$$

m being a positive integer. [*B. U. 1926 Hons.*, I.]

[The result may be obtained directly by integrating, between 0 and $\pi/2$, the identity

$$\sin^{2n}x\cos^{2m}x=\sin^{2n}x(1-\sin^2x)^m=\sin^{2n}x-{}_mC_1\sin^{2n+2}x+\ldots]$$

6. $2^n\cos x\cos 2x\cos 2^2x\cos 2^3x\ldots\cos 2^{n-1}x=\sin 2^nx\div\sin x$.

7. $(1-4\sin^2 x)(1-4\sin^2 2x)(1-4\sin^2 2^2x)\ldots(1-4\sin^2 2^{n-1}x)$
$= (4\cos^2 2^n x-1)\div(4\cos^2 x-1)$.

8. Establish Van der Monde's theorem by induction.

U_0, U_1, U_2, U_3,... are a set of numbers such that $U_0=1$, $U_1=\cos x$ and any three consecutive numbers U_n, U_{n+1}, U_{n+2} are connected by $U_n-2\cos x\, U_{n+1}+U_{n+2}=0$. Prove that $U_r=\cos rx$.

9·7. In some series, only the first few terms are known and neither the general term nor the law of formation of the series is given. In such cases, we employ what is known as the **method of finite differences**, to discover the simplest of the laws that answer to the known terms. Consider the series of terms

$$T_1,\ T_2,\ T_3,\ldots \tag{1}$$

From this series, we derive another series, T_2-T_1, T_3-T_2, T_4-T_3,... the rth term of which is obtained by subtracting the rth term from the $r+1$th term of the preceding series (1). The terms of the new series are denoted by

$$\Delta T_1,\ \Delta T_2,\ \Delta T_3,\ldots \tag{2}$$

The series (2) constitutes what we call the *first order of differences*.

From the series (2), we derive similarly *the second order of differences*.

$$\Delta T_2-\Delta T_1,\ \Delta T_3-\Delta T_2,\ \Delta T_4-\Delta T_3,\ldots\ldots$$

and denote them by

$$\Delta^2 T_1,\ \Delta^2 T_2,\ \Delta^2 T_3,\ldots \tag{3}$$

In this way, we can successively build up higher *orders of differences*.

Example. From the series of numbers

$$1,\ 4,\ 9,\ 16,\ 25,\ \ldots,\ n^2,\ldots$$

the following orders of differences are derived;

3, 5, 7, 9, ... $2n+1$...
2, 2, 2, ... 2...
0 0 0...

In this case, the third and higher orders of differences consisting of zeros.

9·71. We now give some general theorems fundamental in the theory of finite differences.

Prove that

$$T_{n+1} = T_1 + {}_nC_1 \Delta T_1 + {}_nC_2 \Delta^2 T_1 + {}_nC_3 \Delta^3 T_1 + \ldots\ldots$$

We have $T_2 - T_1 = \Delta T_1$;

$$\therefore \quad T_2 = T_1 + \Delta T_1. \qquad (1)$$

Similarly $\Delta T_2 = \Delta T_1 + \Delta^2 T_1. \qquad (1')$

Adding up (1) and (1'), we have

$$T_2 + \Delta T_2 = T_1 + 2\Delta T_1 + \Delta^2 T_1.$$

but $T_3 = T_2 + \Delta T_2 \quad \therefore \quad T_3 = T_1 + 2\Delta T_1 + \Delta^2 T_1. \qquad (2)$

Now starting from the first order of differences instead of the given series, we have similarly

$$\Delta T_3 = \Delta T_1 + 2\Delta^2 T_1 + \Delta^3 T_1 \qquad (2')$$

Adding (2) and (2') and remembering that $T_4 = T_3 + \Delta T_3$, we have $T_4 = T_1 + 3\Delta T_1 + 3\Delta^2 T_1 + \Delta^3 T_1$.

We thus see that the theorem is true for values $n = 2, 3, 4$. Assume that the theorem holds for a particular value r of n so that we have $T_{r+1} = T_1 + {}_rC_1 \Delta T + {}_rC_2 \Delta^2 T_1 + \ldots$

and $\Delta T_{r+1} = \Delta T_1 + {}_rC_1 \Delta^2 T_1 + {}_rC_2 \Delta^3 T_1 + \ldots$

Adding these two relations and remembering that

$$T_{r+2} = T_{r+1} + \Delta T_{r+1}$$

we have $T_{r+2} = T_1 + ({}_rC_1 + 1)\Delta T_1 + ({}_rC_1 + {}_rC_2)\Delta^2 T_1 + \ldots\ldots$

$$= T_1 + {}_{r+1}C_1 \Delta T_1 + {}_{r+1}C_2 \Delta^2 T_1 + \ldots .$$

Thus we find that the theorem is true for the next value $r+1$ of n ; and so by the method of induction the theorem is true for all values of n.

9·72. *Prove that*

$$S_n = {}_nC_1 T_1 + {}_nC_2 \Delta T_1 + {}_nC_3 \Delta^2 T_1 + \ldots\ldots .$$

We have by § 9·71

$$T_n = T_1 + \frac{(n-1)}{1!} \Delta T_1 + \frac{(n-1)(n-2)}{2!} \Delta^2 T_1 + \ldots .$$

where $T_1, \Delta T_1, \ldots$ are independent of n and hence by cor. 4, §9·2

we have $S_n = nT_1 + \frac{n(n-1)}{2!} \Delta T_1 + \frac{n(n-1)(n-2)}{3!} \Delta^2 T_1 + \ldots$

9·73. *Prove that*

$$\Delta^n T_1 = T_{n+1} - {}_nC_1 T_n + {}_nC_2 T_{n-1} - \ldots + (-1)^n {}_nC_n T_1 .$$

By definition, we have

$$\Delta T_1 = T_2 - T_1 \text{ and } \Delta T_2 = T_3 - T_2:$$

$$\therefore \quad \Delta^2 T_1 = \Delta T_2 - \Delta T_1$$

$$= (T_3 - T_2) - (T_2 - T_1)$$

$$= T_3 - 2T_2 + T_1.$$

Similarly

$$\Delta^2 T_2 = T_4 - 2T_3 + T_2$$

$$\therefore \quad \Delta^3 T_1 = \Delta^2 T_2 - \Delta^2 T_1$$

$$= (T_4 - 2T_3 + T_2) - (T_3 - 2T_2 + T_1)$$

$$= T_4 - 3T_3 + 3T_2 - T_1.$$

The proof may be completed by the method of mathematical induction.

Example. 1. Sum, to n terms, the series whose nth term is $n(n^2+1)$.

The terms of the series are

$$2, 10, 30, 68, 130, \ldots n(n^2+1) \ldots$$

and the orders of differences are

$$8, 20, 38, 62, \ldots 3n^2+3n+2 \ldots$$

$$12, 18, 24 \ldots 6n+6 \ldots$$

$$6, 6, \ldots 6 \ldots$$

Hence by § 9·72, the sum is

$${}_nC_1 . 2 + {}_nC_2 8 + {}_nC_3 12 + {}_nC_4 6$$

or

$$\tfrac{1}{4} n(n+1)(n^2+n+2).$$

9·74. *If the r th order of differences with respect to a series consists of equal terms, then the nth term of the series is a polynomial in n of the rth degree; conversely if the nth term is a polynomial of the rth degree the rth order of differences with respect to it consists of equal terms.*

The first part follows immediately from § 9·71, for,

$$T_n = T_1 + {}_{n-1}C_1 \Delta T_1 + {}_{n-1}C_2 \Delta^2 T_1 + \ldots + {}_{n-1}C_r \Delta^r T_1.$$

as the higher orders of differences consist of zeros.

To prove the converse theorem, suppose

$$T_n = a_0 n^r + a_1 n^{r-1} + a_2 n^{r-2} + \ldots + a_r$$

then $$\triangle T_n = T_{n+1} - T_n = a_0 (n+1)^r + a_1 (n+1)^{r-1} + \ldots - [a_0 n^r + a_1 n^{r-1} + \ldots]$$

which is a polynomial of degree $r-1$. Applying the same argument, we can prove that $\triangle^2 T_n$ or $\triangle T_{n+1} - \triangle T_n$ is a polynomial of degree $r-2$. In this way we can prove that $\triangle^r T_n$ is of zero degree.

Example. Sum the series $1+2+5+12+27+58+$ to n terms.

The orders of differences are

1, 3, 7, 15, 31 . . .

2, 4, 8, 16, . . .

We see that the 2nd order of differences consists of terms in G. P.; so that $\triangle^2 T_n = \triangle T_{n+1} - \triangle T_n = 2^n$.

Putting $n = 1, 2, 3 \ldots\ldots n$ in this relation and adding up the equations, we have

$$\triangle T_{n+1} - \triangle T_1 = 2^{n+1} - 2 \text{ or } \triangle T_{n+1} = 2^{n+1} - 1.$$

Now $$\triangle T_{n+1} = T_{n+2} - T_{n+1} = 2^{n+1} - 1.$$

Putting $n = 0, 1, 2, 3, \ldots n-1$ in this, and adding the n relations, we get

$$T_{n+1} - T_1 = 2^{n+1} - 2 - n \text{ or } T_{n+1} = 2^{n+1} - n - 1.$$

$\therefore$ the sum of the series $= 2^{n+1} - 2 - \frac{1}{2} n(n+1)$.

Examples. IX.

Sum the following series to n terms by the method of differences :

1. $3+8+15+24+35+\ldots$.

2. $-1+0+3+8+15+\ldots$.

3. $6+24+60+120+210+\ldots$.

4. $-8-5+0+8+17+31\ldots$

5. $3+5+8+10+13+15+18+\ldots$.

9·8. The following examples should be carefully studied as they illustrate two important methods of summation.

Example 1. Prove that $c_0 - c_1 + c_2 - \ldots (-1)^r c_r = (-1)^r (n-1)_r \div r!$, where c_r is the coefficient x^r in $(1+x)^n$.

We have $(1-x)^n = c_0 - c_1 x + c_2 x^2 - c_3 x^3 + \ldots + (-1)^n c_n x^n$

and $(1-x^{m+1})/(1-x) = 1 + x + x^2 + \ldots + x^m$, where $m > r$.

Multiplying the identities, we have

$$(c_0 - c_1 x + c_2 x^2 - \ldots + (-1)^r c_r x^r + \ldots)(1 + x + x^2 + x^3 + \ldots)$$
$$= (1-x)^{n-1}(1-x^{m+1}) = (1 - {}_{n-1}C_1 x + {}_{n-1}C_2 x^2 - \ldots)(1-x^{m+1})$$

The desired result is obtained at once by equating the coefficients of x^r on both sides of this identity.

This example may also be solved by the method of § 9·4 or more easily, by using the formula $c_p = {}_{n-1}C_p + {}_{n-1}C_{p-1}$

Example. 2. $(1 + x + x^2 + x^3)^n = a_0 + a_1 x + a_2 x^2 + \ldots + a_{3n} x^{3n}$

and L, M, N are respectively equal to $a_0 + a_3 + a_6 + a_9 + \ldots$, $a_1 + a_4 + a_7 + \ldots$, $a_2 + a_5 + a_8 + \ldots$, prove that $3L - 2 = 3M + 1 = 3N + 1 = 4^n$

Putting $x = w$ in the given identity where w is a complex cube root of unity, and using the properties of the cube root (§ 7·4) we have

$$1 = L + wM + w^2 N. \qquad (1)$$

Similarly, putting $x = w^2$ which is the other complex cube root, we have

$$1 = L + w^2 M + wN. \qquad (2)$$

Further, putting $x = 1$, we have

$$4^n = L + M + N. \qquad (3)$$

Adding (1), (2), (3) and remembering that $1 + w + w^2 = 0$ we have

$$4^n + 2 = 3L. \qquad (4)$$

Now multiply (1), (2), (3) by w, w^2, 1 and then add them; we thus have

$$4^n - 1 = 3N. \qquad (5)$$

Similarly, multiplying (1), (2), (3) by w^2, w, 1 and then adding them, we have

$$4^n - 1 = 3M. \qquad (6)$$

The required results are obvious from (4), (5), (6).

For a detailed study of the methods of this chapter, the student may read Chrystal's *Algebra*, part II, chapter XXXI.

Examples. IX. j.

1. In the notation of §9·22, prove the following identities:

$$({}_1a_0+{}_1a_1x+{}_1a_2x^2+\ldots+{}_1a_{n-1}x^{n-1})(1-x)^2=1-(n+1)x^n+nx^{n+1}.$$

$$({}_2a_0+{}_2a_1x+{}_2a_2x^2+\ldots+{}_2a_{n-1}x^{n-1})(1-x)^3$$
$$=1-\tfrac{1}{2}(n+1)(n+2)x^n+n(n+2)x^{n+1}-\tfrac{1}{2}n(n+1)x^{n+2}.$$

Generally

$$({}_ra_0+{}_ra_1x+{}_ra_2x^2+\ldots+{}_ra_{n-1}x^{n-1})(1-x)^{r+1}=1-x^nf(x)$$

where $f(x)$ is a polynomial of the rth degree.

2. If $(1+x+x^2)^n=a_0+a_1x+a_2x^2+\ldots+a_{2n}x^{2n}$ prove that

$$a_r.\ {}_nC_0-a_{r-1}.\ {}_nC_1+a_{r-2}.\ {}_nC_2-a_{r-3}.\ {}_nC_3+\ldots+(-1)^ra_0.\ {}_nC_r$$

is either zero or ± 1 according as r is not or is, a multiple of 3.

3. Prove that

$$\frac{1}{n!}+\frac{1}{4!\,(n-4)!}+\frac{1}{8!\,(n-8)!}+\ldots=\frac{2^{n-2}}{n!}\left[1+2\cos^n\tfrac{1}{3}\pi\cos\tfrac{1}{3}n\pi\right].$$

4. If $l=a_0+a_3x^3+a_6x^6+\ldots$

$$m=a_1x+a_4x^4+a_7x^7+\ldots$$
$$n=a_2x^2+a_5x^5+a_8x^8+\ldots$$

where $(1+x)^n=a_0+a_1x+a_2x^2+a_3x^3+\ldots+a_nx^n$ prove that

$$l^3+m^3+n^3-3\,l\,m\,n=a_0+a_1x^3+a_2x^6+a_3x^9+\ldots+a_nx^{3n}.$$

5. If l, m, n are respectively equal to

$$a_0+a_3+a_6+\ldots,\ a_1+a_4+a_7+\ldots,\ a_2+a_5+a_8+\ldots$$

where $(1+x+x^2)^n=a_0+a_1x+a_2x^2+\ldots+a_{2n}x^{2n}$, prove that

$$l=m=n=3^n.$$

6 *If w is a special sth root of unity and*

$$f(x)\equiv c_0+c_1x+c_2x^2+\ldots+c_nx^n$$

prove that

$$f(x)+w^{-r}f(xw)+w^{-2r}f(xw^2)+\ldots+w^{-rs+r}f(xw^{s-1})$$
$$=s(c_rx^r+c_{r+s}x^{r+s}+c_{r+2s}x^{r+2s}+\ldots)$$

r being zero or a positive integer less than s.

Examples. IX. k.

Sum the following series to n terms:

1. $\dfrac{1}{2!}+\dfrac{2}{3!}+\dfrac{3}{4!}+\dfrac{4}{5!}+\ldots+\dfrac{n}{(n+1)!}$

2. $\dfrac{1.2^1}{4.2^2-1}+\dfrac{3.2^2}{4.3^2-1}+\dfrac{5.2^3}{4.4^2-1}+\ldots.$

3. $\dfrac{1+1^2}{1+1^2+1^4}-\dfrac{1+2^2}{1+2^2+2^4}+\dfrac{1+3^2}{1+3^2+3^4}-\dots$.

4. $2^{-1}\tan 2^{-1}x+2^{-2}\tan 2^{-2}x+2^{-3}\tan 2^{-3}x+\dots$.

5. $\dfrac{2}{1.3.5.7}+\dfrac{3}{3.5.7.9}+\dfrac{4}{5.7.9.11}+\dots$

6. $\dfrac{\cos 2x}{\cos^2 2x\cos^2 x-\sin^2 2x\sin^2 x}-\dfrac{\cos 4x}{\cos^2 4x\cos^2 x-\sin^2 4x\sin^2 x}$

$+\dfrac{\cos 6x}{\cos^2 6x\cos^2 x-\sin^2 6x\sin^2 x}-\dots$.

7. $\cos x.\cos 1^2x-\cos 2x.\cos 2^2x+\cos 3x.\cos 3^2x-\dots$.

8. $\cos^3 x-3^{-1}\cos^3 3x+3^{-2}\cos^3 3^2x-3^{-3}\cos^3 3^3x+\dots$.

9. $\dfrac{2.1}{1}+\dfrac{2.4.2}{1.3}+\dfrac{2.4\,6.3}{1.3.5}+\dfrac{2.4.6.8.4}{1.3\,5.7}+\dots$.

10. $\dfrac{2.1^2}{1}+\dfrac{2.4.2^2}{1.3}+\dfrac{2\,4.6.3^2}{1.3.5}+\dfrac{2.4\,6.8.4^2}{1.3.5.7}+\dots$.

11. $\dfrac{2}{1.5}+\dfrac{2.4}{1.3.7}+\dfrac{2.4\,6}{1\,3.5.9}+\dots$.

12. $1^3+3^3+5^3+7^3+\dots$.

13. Put $-1,-s-1,n-1$ for m, n, r respectively in the identity of § 5·213, multiply the result by $(s+1)!\div(n-1)!$ and thus obtain

$(s+1)[1.2.3\dots s+2.3.4\dots(s+1)+\dots+n(n+1)(n+2)\dots(n+s-1)]$

$=n(n+1)(n+2)\dots(n+s)$.

Deduce that

$(s+1)$ [$(n+1)(n+2)\dots(n+s)+(n+2)(n+3)\dots(n+s+1)+\dots$ to m terms] $=(m+n)(m+n+1)\dots(m+n+s)-n(n+1)(n+2)\dots(n+s)$.

Now apply the argument of § 5·213 to this result and establish the identity of § 9·2.[216] [*c. f* § 9.2]

13.[a] Put $-m,-1,r$ for m, n, r in the identity of §5·213, divide the result by $r!$ and thus obtain

$$1+\frac{m}{1}+\frac{m(m+1)}{1.2}+\frac{m(m+1)(m+2)}{1.2.3}+\dots\text{to } r+1\text{terms}$$

$$=\frac{(m+1)(m+2)\dots(m+r)}{1.2.3\dots.r}.$$

Deduce that

$$\frac{m+r}{r+1}+\frac{(m+r)(m+r+1)}{(r+1)(r+2)}+\frac{(m+r)(m+r+1)(m+r+2)}{(r+1)(r+2)(r+3)}+\ldots$$

$$\text{to } n \text{ terms} = \frac{m+r}{m}\left\{\frac{(m+r+1)(m+r+2)\ldots(m+r+n)}{(r+1)(r+2)\ldots(r+n)}-1\right\}.$$

14. Put $-r$, $-s$, n for m, n, r in the identity of § 5·213, and divide the result by $n!$ and thus obtain

$$\frac{s(s+1)(s+2)\ldots(s+n-1)}{n!}+\frac{s(s+1)\ldots(s+n-2)}{(n-1)!}\cdot\frac{r}{1!}+\ldots$$

$$+\frac{s}{1!}\cdot\frac{r(r+1)(r+2)\ldots(r+n-2)}{(n-1)!}+\frac{r(r+1)\ldots(r+n-1)}{n!}$$

$$=\frac{(r+s)(r+s+1)\ldots(r+s+n-1)}{n!}.$$

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15. In the notation of § 9·22, prove that

$$s a_n \cdot r a_0 + s a_{n-1} \cdot r a_1 + s a_{n-2} \cdot r a_2 + \ldots + s a_1 \cdot r a_{n-1} + s a_0 \cdot r a_n = r+s+1 a_n.$$

16. Prove that $A_n \cdot A_0 + A_{n-1} \cdot A_1 + A_{n-2} A_2 + \ldots + A_0 A_n = 1$

where $A_r = [\,1.3.5\ldots(2r-1)\,] \div [\,2.4.6\ldots 2r\,]$

[Put $r=s=-\frac{1}{2}$ in the example 14., above].

17. In the notation of § 5·213, prove that

$$r_r \cdot (n+s)_s + (r+1)_r \cdot (n+s-1)_s + (r+2)_r \cdot (n+s-2)_s + \ldots$$

$$\ldots + (r+n-1)_r (s+1)_s + (r+n)_r s_s = r!\, s!\, (n+r+s+1)_{r+s+1} \div (r+s+1)!$$

e.g. $1.(n+1)+2.\,n+3.\,(n-1)+\ldots$ to $n+1$ terms

$$=\tfrac{1}{6}(n+1)(n+2)(n+3).$$

$1.2(n+1)+2.3.n+3,4(n-1)+\ldots$ to $n+1$ terms

$$=\tfrac{1}{12}(n+1)(n+2)(n+3)(n+4).$$

$1.2(n+1)(n+2)+2.3.n\,(n+1)+3.4.(n-1)n+\ldots$ to $n+1$ terms

$$=\tfrac{1}{30}(n+1)(n+2)(n+3)(n+4)(n+5).$$

18. Prove that the first n terms in the expansions of $(1+2x+3x^2+\ldots+nx^{n-1})^3$ and $(1+3x+6x^2+\ldots\tfrac{1}{2}n(n+1)x^{n-1})^2$ are the same.

19. *In the notation of §5·213 prove that*

$$r_r - {}_nC_1 (r+1)_r + {}_nC_2 (r+2)_r - \ldots \text{ to } n+1 \text{ terms} = 0$$

where n, r are positive integers ($n>r$).

[Put n, $-r-1$, n for m, n, r respectively in the identity of § 5·213 and then multiply the result by $r! \div n!$.]

e. g. $1.2-3\ 2.3+3.3.4-4.5=0$

and if $n>2$ $\quad 1.2-{}_nC_1.\ 2.3+{}_nC_2\ 3.4-{}_nC_3\ 4.5+\ldots=0.$

20. If n and r are positive integers ($n>r$) and if $f($ polynomial in p of the rth degree, prove that

$$0=f(1)-{}_nC_1\ f(2)+{}_nC_2 f(3)-\ldots\text{to } n+1 \text{ terms}$$

$$0=f(n+1)-{}_nC_1 f(n)+{}_nC_2{}^{\cdot}(n-1)-\ldots\text{to } n+1 \text{ terms.}$$

e. g. $\quad (a+d)-3(a+2d)+3(a+3d)-(a+4d)=0;$

$$(a+d)^2-3(a+2d)^2+3(a+3d)^2-(a+4d)^2=0;$$

$$(a+d)^3-4(a+2d)^3+6(a+3d)^3-4(a+4d)^3+(a+5d)^3=0;$$

[$f(p)$ can be transformed into $a_0+a_1p+a_2p(p+1)+\ldots$(§ 9·201).]

21. If n is a positive integer greater than 3 prove that

$$n^3-{}_nC_1(n-1)^3+{}_nC_2(n-2)^3-{}_nC_3(n-3)^3+\ldots=0\ .$$

[*B. U., 1923 Hons., I.*]

22. When n is a positive integer, prove that

$n^{n-1}-{}_nC_1(n-1)^{n-1}+{}_nC_2(n-2)^{n-1}\ldots=0$. [*B. U., 1919 Hons., I.*]

23. *Prove that*

$$(-1)^n n_n=n_n-{}_nC_1(n+1)_n+{}_nC_2(n+2)_n-\ldots$$

$$(-1)^n(n+1)_n(n+1)_1=(n+1)_{n+1}$$
$$-{}_nC_1(n+2)_{n+1}+{}_nC_2(n+3)_{n+1}-\ldots$$

$$(-1)^n(n+r)_n(n+r)_r=(n+r)_{n+r}-$$
$$-{}_nC_1(n+r+1)_{n+r}+{}_nC_2(n+r+2)_{n+r}-\ldots$$

[To obtain the last result put, $n,-n-r-1,n$ for m,n,r in the identity of §5·213, and then multiply the result by $(n+r)!\div n!$.]

24. Prove that when n is an even integer the value of the expression

$$n^p-{}_nC_1(n-2)^p+{}_nC_2(n-4)^p-\ldots\ .$$

is zero if $p<n$ and find the value when $p=n$.

[*B. U., 1920, Hons , I.*]

25. When n is a positive integer, prove that

$n^n-{}_nC_1(n-1)^n+{}_nC_2(n-2)^n-\ldots=n!$. [*B. U.; 1914, Hons.* I.]

26. Shew (by induction or otherwise) that if n and k are positive integers, then

$$f_{n,k} \equiv x^{n-k}C_1(x+y)^{n+k}C_2(x+2y)^{n-k}C_3(x+3y)^{n}+\dots$$

contains no term in x of degree higher than $n-k$; and deduce that

$$f_{n,n} = (-1)^n n! y^n.$$

[*B. U. 1922, Hons*, I.]

27. Prove that

$$(1-x)^2(1+2x+3x^2+\dots+nx^{n-1}) = 1-(n+1)x^n+nx^{n+1};$$

$$(1-x)^3(1^2+2^2x+3^2x^2+\dots+n^2x^{n-1})$$
$$= 1+x-(n+1)^2x^n+(2n^2+2n-1)x^{n+1}-n^2x^{n+2};$$

and generally

$$(1-x)^{r+2}(1^{r+1}+2^{r+1}x+3^{r+1}x^2+\dots+n^{r+1}x^{n-1}) = P_r(x)+x^nQ_{r+1}(x),$$

where $P_r(x)$ and $Q_{r+1}(x)$ are polynomials in x of degrees r, $r+1$ respectively.

28. Prove that

$$(p+1)^{r+1}-{}_{r+2}C_1p^{r+1}+{}_{r+2}C_2(p-1)^{r+1}-\dots+(-1)^p{}_{r+2}C_p1^{r+1}$$
$$= (r-p+1)^{r+1}-{}_{r+2}C_1(r-p)^{r+1}+{}_{r+2}C_2(r-p-1)^{r+1}\dots$$
$$+{}_{r+2}C_{r-p}(-1)^{r-p}1^{r+1}.$$

29. Prove that

$$1-\frac{n^2}{1^2}+\frac{n^2(n^2-1^2)}{1^2.2^2}-\frac{n^2(n^2-1^2)(n^2-2^2)}{1^2.2^2.3^2}+\dots\text{to } r+1\text{terms}$$
$$=\frac{(1^2-n^2)(2^2-n^2)\dots(r^2-n^2)}{1^2.2^2.3^2\dots r^2}.$$

30. Prove that

$$1+\frac{a(1-a)}{1!2!}+\frac{a(1^2-a^2)(2-a)}{2!3!}+\frac{a(1^2-a^2)(2^2-a^2)(3-a)}{3!4!}+\dots$$
$$\text{to } n+1 \text{ terms} = \frac{(1+a)(2^2-a^2)(3^2-a^2)\dots(n^2-a^2)(n+1-a)}{n!(n+1)!}.$$

---

31. Determine the law of the series

1, 2, 29, 130, 377, 866, 1717,...

using the method of differences or other and find the sum to n terms.

[*B. U., 1920, Hons., I*]

32. Prove that

$(1-4\sin^2 x)(1-4\sin^2 3x)(1-4\sin^2 3^2x)\ldots(1-4\sin^2 3^{n-1}x) = \cos 3^n x \sec x$,

and that $(1-4\sin^2 3^{-1}x)(1-4\sin^2 3^{-2}x)(1-4\sin^2 3^{-3}x)\ldots(1-4\sin^2 3^{-n}x)$

$$= \cos x \sec 3^{-n}x.$$

33. Sum the series to n terms

$$2\tan x \sec 2x + 2^2 \tan 2x \sec 2^2 x + 2^3 \tan 2^3 x \sec 2^3 x + \ldots$$

MISCELLANEOUS EXAMPLES I

1. If $(1+x+x^2+\ldots+x^r)^m = a_0 + a_1 x + a_2 x^2 + \ldots + a_{mr} x^{mr}$

then $na_0 + (n+1)a_1 + (n+2)a_2 + \ldots + (n+mr)a_{mr}$

$$= \tfrac{1}{2}(mr+2n)(r+1)^m.$$

2. Prove that

$$\begin{vmatrix} a & -1 & 0 & 0 \\ b & 1 & -1 & 0 \\ c & 0 & 1 & -1 \\ d & 0 & 0 & 1 \end{vmatrix} = a+b+c+d.$$

3. Prove that

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ x & x & a & x \\ x & b & x & x \\ c & x & x & x \end{vmatrix} = (a-x)(b-x)(c-x).$$

4. In a triangle ABC of given species, A is fixed and B moves on a given circle; prove that C also moves on a fixed circle.

5. Shew that the necessary and sufficient condition that the product of one of the roots of

$$ax^2+bx+c=0 \text{ and one of } a_1x^2+b_1x+c_1=0$$

should be equal to the product of the remaining roots, is

$$ac(b^2-2ac)^{-1} = a_1c_1(b_1^2-2a_1c_1)^{-1}.$$

6. p is a root of $x^2+ax+b=0$, and q of $x^2+cx+d=0$, prove that pq is a root of $x^4-acx^3+(a^2d-2bd+bc^2)x^2-abcdx+b^2d^2=0$.

7. Prove that $2^{n-1}\,n!$ is equal to

$$n(n^2-1^2)(n^2-3^2)(n^2-5^2)\ldots[n^2-(n-2)^2]$$

or

$$n^2(n^2-2^2)(n^2-4^2)(n^2-6^2)\ldots[n^2-(n-2)^2]$$

according as n is odd or even.

8. Sum the series to n terms

$$\tan^{-1}\frac{2}{1-1^2+1^4}+\tan^{-1}\frac{4}{1-2^2+2^4}+\tan^{-1}\frac{6}{1-3^2+3^4}+\ldots.$$

*9. If w is a complex 7th root of unity, prove that $w+w^6+w^4$, $w^3+w^5+w^2$ are the roots of the equation $x^2+x+2=0$

Deduce that $\sin\frac{2}{7}\pi+\sin\frac{4}{7}\pi+\sin\frac{8}{7}\pi=\frac{1}{2}\sqrt{7}$.

*10. If w is a complex 11th root of unity, prove that

$$w+w^3+w^4+w^5+w^9,\quad w^2+w^6+w^7+w^8+w^{10}$$

satisfy $x^2+x+3=0$. Deduce that $\frac{1}{2}\sqrt{11}$ is equal to

$$\sin\frac{2\pi}{11}+\sin\frac{6\pi}{11}+\sin\frac{8\pi}{11}+\sin\frac{10\pi}{11}+\sin\frac{18\pi}{11}.$$

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11. If r is a positive integer, prove that the first $n+1$ terms in the expansion of

$$\left\{1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots+\frac{x^n}{n!}\right\}^r$$

are given by

$$1+\frac{rx}{1!}+\frac{r^2x^2}{2!}+\frac{r^3x^3}{3!}+\ldots+\frac{r^nx^n}{n!}.$$

12. Shew that in the usual notation for Δ

$$\begin{vmatrix} s & a & a & a \\ a & s & b & b \\ b & b & s & c \\ c & c & c & s \end{vmatrix} = 3\Delta^2$$

$$= \begin{vmatrix} s & a-s & 0 & 0 \\ a & s-a & b-a & 0 \\ b & 0 & s-b & c-s \\ c & 0 & 0 & s-c \end{vmatrix} = \overline{s-a}\,\overline{s-b}\,\overline{s-c} \begin{vmatrix} 3s & a & b & c \\ a & 1 & -1 & 0 \\ b & 0 & 1 & -1 \\ c & 0 & 0 & 1 \end{vmatrix}$$

13. Shew that

$$\begin{vmatrix} (n-2)a & a+nb & a+nc \\ b+na & (n-2)b & b+nc \\ c+na & c+nb & (n-2)c \end{vmatrix} = 3n(a+b+c)(bc+ca+ab).$$

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* The two examples 9, 10 belong to the interesting Theory of Cyclotomy [G. B. Mathews' *Theory of Numbers*, Part I, Chap. III.[

14. AB and CD ard divided at L and M in the same ratio and LM is divided at N in a fixed ratio. Prove that the locus of N is a st. line.

15. Shew that the equation

$$25x^4-298x^3+576x^2-281x-26=0$$

has one and only one real negative root and find its value correct to three places of decimals. [*B. U. M. A. 1924.*]

16. The condition that two of the roots of

$$(ax^2+2bx+c)^2=(dx+e)^2$$

should be equal, is $\begin{vmatrix} a & b-d & \\ b-d & c-e & \end{vmatrix}\begin{vmatrix} a & b+d \\ b+d & c+e \end{vmatrix}+\begin{vmatrix} a & 2b & c \\ 2d & e & \end{vmatrix}=0$

$$[(b-d)^2-a(c-e)][(b+d)^2-a(c+e)][ae^2-4d(bc-cd)]=0.$$

17. S_n denotes the sum to n terms of the series

$$\frac{a}{b}+\frac{a(a+x)}{b(b+x)}+\frac{a(a+x)(a+2x)}{b(b+x)(b+2x)}+\dots$$

and S_n' denotes the sum of the reciprocals of the terms of this series; prove that

$$[S_n(a-b+x)+a][S_n'(b-a+x)+b]=(a+nx)(b+nx).$$

18. If $(1+x)^n=c_0+c_1x+c_2x^2+\dots+c_nx^n$, prove that

$$(r+1)c_0-rc_1+(r-1)c_2-(r-2)c_3+\dots+(-1)^rc_r=(-1)^r{}_{n-2}C_r;$$

$$1c_0-2c_1+3c_2-\dots+(r+1)(-1)^rc_r=(-1)^r[{}_{n-1}C_r+n.\ {}_{n-2}C_{r-1}].$$

Deduce corresponding results true for all values of n.

19. Prove the identity.

$$[(\cos x-\sin x\tan a)+i(\cos x\tan a+\sin x)]^n\cos ny$$
$$-i[(\cos x-\sin x\tan a)-i(\cos x\tan a+\sin x)]^n\sin ny$$
$$=\sec^n a[\cos n(x+y+a)+i\sin n(x+y-a)]$$

20. If $\cos a_1+\cos a_2+\cos a_3+\cos a_4=0$

$$=\sin a_1+\sin a_2+\sin a_3+\sin a_4$$

prove that for all integral values of r

$$\cos(2r-1)a_1+\cos(2r-1)a_2+\cos(2r-1)a_3+\cos(2r-1)a_4=0$$

and that $\sin(2r-1)a_1+\sin(2r-1)a_2+\sin(2r-1)a_3+\sin(2r-1)a_4=0$.

21. If $(1+x)^n = c_0 + c_1x + c_2x^2 + \ldots + c_rx^r + \ldots + c_nx^n$ prove that the first $r+1$ terms in the expansion of

$$(c_0 + c_1x + c_2x^2 + \ldots + c_rx^r)^m$$

are given by $1 + {}_mC_1x + {}_mC_2x^2 + {}_mC_3x^3 + \ldots + {}_mC_rx^r$.

22. Prove that $(x-1)^4$ is equal to

$$\begin{vmatrix} x^3 & x^2 & x & 1 \\ 3x^2 & 2x & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 3 & 2 & 1 & 0 \end{vmatrix}.$$

23. Prove that

$$\begin{vmatrix} x^6 & x^2 & x & 1 \\ 6x^5 & 2x & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 6 & 2 & 1 & 0 \end{vmatrix}$$

$$= 2(x-1)^4[2(x-1)^3 + 9(x-1)^2 + 15(x-1) + 10].$$

24. $\triangle ABC$ is similar to $\triangle PQR$ and the joins of the corresponding vertices AP, BQ, CR are divided at L, M, N in the same ratio. Prove that $\triangle LMN$ is similar to $\triangle ABC$.

25. The two equations

$$2x^4 + 6x^3 + x^2 - 4x + 1 = 0$$

and

$$3x^5 - 9x^4 + 13x^3 - 11x^2 + 5x - 1 = 0$$

remain unchanged after transforming them by writing $x \div (2x-1)$ for x: solve them.

26. If a, b, c, d be the roots of the equation $x^4 - px^3 + qx^2 + r = 0$ form the equation whose roots are bcd, cda, dab, abc.

Also prove that the equation whose roots are

$$cd+db+bc,\ cd+da+ac,\ da+ab+bd,\ ab+bc+ca.$$

is $y^4 - 2qy^3 + (q^2+2r)y^2 + (p^2-2q)ry + r^2 = 0$

[*B. U. M. A. 1924*]

27. Prove that the sum to n term of the series

$$\frac{1}{x} + \frac{1!y}{x(x+y)} + \frac{2!y^2}{x(x+y)(x+2y)} + \frac{3!y^3}{x(x+y)(x+2y)(x+3y)} + \ldots$$

$$= \frac{1}{y-x}\left\{\frac{n!y^n}{x(x+y)(x+2y)\ldots[x+(n-1)y]} - 1\right\}.$$

28. Prove that the sum to n terms of the series

$$x^{n-1}+\frac{(a+x)x^{n-2}}{1!}+\frac{(a+x)(a+2x)x^{n-3}}{2!}$$
$$+\frac{(a+x)(a+2x)(a+3x)x^{n-4}}{3!}+\dots$$
$$=\frac{(a+2x)(a+3x)(a+4x)\dots(a+nx)}{(n-1)!}$$

29. Prove that in the usual notation

$$\Sigma a^n b^n c^n \cos nA = p^n - {}_nC_2 4^2\triangle^2 . p^{n-2} + {}_nC_4 4^4\triangle^4 . p^{n-4} - \dots$$

and $$\Sigma a^n b^n c^n \sin nA = {}_nC_1 4\triangle p^{n-1} - {}_nC_3 4^3\triangle^3 p^{n-3} + \dots$$

where $p = b^2+c^2-a^2$ and $\triangle =$ the area of $\triangle ABC$.

30. Shew that if

$$m_r = m^2(m^2-1^2)(m^2-2^2)\dots[m^2-(r-1)^2]$$

and $c_0, c_1, c_2, \dots c_{2r}$ are coefficients in the expansion of $(1+x)^{2r}$,

$$c_0 m_r + c_2 m_{r-1} n_1 + c_4 m_{r-2} n_2 + \dots + c_{2r} n_r = \tfrac{1}{2}[(m+n)_r + (m-n)_r]$$

where $m > n$. [*B. U., 1927 Hons.*, II]

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31. Prove that

$$(1+x+x^2)(1+x^3+x^6)(1+x^9+x^{18})(1+x^{27}+x^{54})\dots\text{to } n \text{ factors}$$
$$= 1+x+x^2+x^3+\dots\text{to } 3^n \text{ terms.}$$

32. Given 3^0, 3^1, 3^2,...3^{n-1} lbs. weights; prove that it is possible to weigh uniquely any integral number oi lbs, froun 1 to $\frac{1}{2}(3^n-1)$. In particular show how to weigh 47lbs with 3^0, 3^1, 3^2, 3^3, 3^4 lbs. weights.

33. Prove that

$$(1+2\cos y)(1+2\cos 3y)(1+2\cos 3^2 y)\dots\text{to } n \text{ factors}$$
$$= 2\cos\tfrac{1}{2}(3^n-1)y + 2\cos\tfrac{1}{2}(3^n-3)y + 2\cos\tfrac{1}{2}(3^n-5)y + \dots .$$

34. Sum the series

$$\frac{\sin y}{1+2\cos y}+\frac{3\sin 3y}{1+2\cos 3y}+\frac{3^2\sin 3^2 y}{2+2\cos 3^2 y}+\dots \text{ to } n \text{ terms}$$

35. Shew that $2abc\,(a+b+c)^3$ is equal to

$$\begin{vmatrix} (b+c)^2 & ab & ac \\ ba & (c+a)^2 & bc \\ ca & cb & (a+b)^2 \end{vmatrix} .$$

36. Prove that $(a-b)^4(x-a)^2(x-b)^2$ is equal to

$$\begin{vmatrix} x^4 & x^3 & x^2 & x & 1 \\ a^4 & a^3 & a^2 & a & 1 \\ 4a^3 & 3a^2 & 2a & 1 & 0 \\ b^4 & b^3 & b^2 & b & 1 \\ 4b^3 & 3b^2 & 2b & 1 & 0 \end{vmatrix}.$$

37. AB, CD are divided at L and M in the same ratio and $\triangle LMN$ is of fixed species. Prove that the locus of N is a st. line

38. Determine a, b, c so that the polynomial

$$2x^3+ax^2+bx+c$$

should be divisible by $(x-1)^3$.

39. If a, b, c are the roots of the equation

$$x^3+3px^2+3qx+r=0$$

and if S_n is equal to $(a+p)^n+(b+p)^n+(c+p)^n$, prove that

$$\tfrac{1}{5}S_5=\tfrac{1}{2}S_2\,\tfrac{1}{3}S_3$$

40. If n is of the form $4m+1$ and $2na=\pi$. prove that

$$(1-\sin 2x\operatorname{cosec} a)(1+\sin 2x\operatorname{cosec} 3a)(1-\sin 2x\operatorname{cosec} 5a)\ldots$$
$$\times[1+\sin 2x\operatorname{cosec}(n-2)a]=(\sin nx-\cos nx)\div(\sin x-\cos x).$$

---

41. If $F(a, x)=1+\dfrac{ax}{1!}+\dfrac{a^2x^2}{2!}+\dfrac{a^3x^3}{3!}+\ldots+\dfrac{a^mx^m}{m!}$

prove that the first $m+1$ terms in the expansion of

$$F(a, x)\times F(b, x)\times F(c, x)\times\ldots\times F(k, x)$$

in ascending powers of x, are given by $F(a+b+c+\ldots k, x)$.

[Apply § 1·41]

42. Prove that

$$\begin{vmatrix} (a+d)^2 & (a+2d)^2 & (a+3d)^2 & (a+4d)^2 \\ (a+2d)^2 & (a+3d)^2 & (a+4d)^2 & (a+5d)^2 \\ (a+3d)^2 & (a+4d)^2 & (a+5d)^2 & (a+6d)^2 \\ (a+4d)^2 & (a+5d)^2 & (a+6d)^2 & (a+7d)^2 \end{vmatrix}=0.$$

43. St. lines AL, BM, CN are drawn equal and parallel to the sides of a triangle. Prove that $\triangle ABC$ and $\triangle LMN$ have a common centroid.

44. Construct a polynomial $P(x)$ of the 5th degree such that $P(x)-1$ and $P(x)+1$ are divisible by $(x-1)^3$ and $(x+1)^3$ respectively.

45. Prove that the equation $(x-p)^{2n}=2nq^{2n-1}x$ has two real roots or none according as $q[p(1-2n)-2np]$ is negative or positive; p and q are given to be real numbers.

46. Solve the equations $a+b+c=0$, $a^2+b^2+c^2=12$ and $a^3+b^3+c^3=27$. [*B. U. 1927 Hons.*, I]

47. Prove that

$$\left\{1-\frac{a_1}{1+a_1}-\frac{a_2}{(1+a_1)(1+a_2)}-\frac{a_3}{(1+a_1)(1+a_2)(1+a_3)}\right.$$
$$\left.-\ldots \text{ to } n \text{ terms}\right\}$$
$$\times\left\{1+\frac{a_1}{1-a_1}+\frac{a_2}{(1-a_1)(1-a_2)}+\frac{a_3}{(1-a_1)(1-a_2)(1+a_3)}\right.$$
$$\left.+\ldots \text{ to } n \text{ terms}\right\}$$
$$=1+\frac{a_1^2}{1-a_1^2}+\frac{a_2^2}{(1-a_1^2)(1-a_2^2)}+\frac{a_3^2}{(1-a_1^2)(1-a_2^2)(1-a_3^2)}$$
$$+\ \ldots \text{ to } n \text{ terms.}$$

48. Prove that

$$2\cos a.\cos a+2\cos a.\cos(a+2b)+2^2\cos a.\cos b.\cos(a+b+2c)$$
$$+2^3\cos a.\cos b.\cos c.\cos(a+b+c+2d)+\ldots \text{to } n \text{ terms}$$
$$=2^n\cos a.\cos b.\cos c\ldots\cos k\cos(a+b+c+\ldots+k),$$

and that $2\cos a\sin a+2\cos a.\sin(a+2b)+2^2\cos a.\cos b.\sin(a+b+2c)$
$+2^3\cos a\cos b.\cos c.\sin(a+b+c+2d)+\ldots$ to n terms
$$=2^n\cos a.\cos b.\cos c.\ldots\cos k\sin(a+b+c+\ldots.).$$

49. If $f(x)\equiv x^n-\frac{n}{1!}x^{n-2}+\frac{n(n-3)}{2!}x^{n-4}-\ldots$

$$+\ (-1)^r\frac{n(n-r-1)(n-r-2)\ldots(n-2r+1)}{r!}x^{n-2r}+\ldots.$$

then $$(x-2)[f_{n+1}(x)-2]\equiv[f_{n+1}(x)-f_n(x)]^2.$$

50. $P_1 P_2 P_3 \ldots P_l$ and $Q_1 Q_2 Q_3 \ldots Q_m$ are two regular polygons not necessarily in one plane, prove that

$$\sum_{s=1}^{s=m} \sum_{r=1}^{r=l} P_r Q_s^2 = lm(a^2+b^2+d^2)$$

where a and b are the radii of circles circumscribing the polygons and d the distance between their centres.

---

51. If $f(x) \equiv 1+x+x^2+x^3+\ldots+x^{p-1}$, prove that

$$\sum_{r=0}^{r=n-1} f(x^m) = 1+x+x^2+x^3+\ldots \text{ to } p^n \cdot 1 \text{ terms.}$$

where $m = p^r$.

52. Prove that $2(a^2+b^2+c^2)^3$ is equal to

$$\begin{vmatrix} 2a^2+b^2+c^2 & ba & ca \\ ab & a^2+2b^2+c^2 & cb \\ ac & bc & a^2+b^2+2c^2 \end{vmatrix}.$$

53. Prove that $2(bc+ca+ab)(a+b+c)^4$ is equal to

$$\begin{vmatrix} (a+b+c)^2-a^2 & -ba & -ca \\ -ab & (a+b+c)^2-b^2 & -cb \\ -ac & -bc & (a+b+c)^2-c^2 \end{vmatrix}.$$

* 54. L, L'; M, M'; N, N' are taken on the sides BC, CA, AB of a triangle so that LL', MM' NN' are proportional to BC, CA AB. Prove that triangles LMN and $L'M'N'$ have a common centroid.

55. Solve the equations $4(x^5+1) = (x-1)^2(x+1)^3$ and
$(x^3-1)(x^2+x+1) = 4(x-1)(x^2-x+1)^2$.

56. A polynomial $f(x)$ is identically equal to $\pm f(2-x)$. Prove that $f(x)$ can be written as an odd or even polynomial in $x-1$.

57. If $f(x) = x^r(x+1)^r(2x+1)^p$ prove that
$$f(x-1) = (-)^p f(-x)$$

---

* Examples 43, 54 can be solved most easily by statical methods.

and that $f(x)-f(x-1)$ is an odd or even polynomial in x according as p is even or odd.

Prove by using the identity

$$x^r(x+1)^r-x^r(x-1)^r=2({}_rC_1x^{2r-1}+{}_rC_3x^{2r-3}+{}_rC_5x^{2r-5}+\ldots)$$

$$n^r(n+1)^r=({}_rC_1\cdot{}_{2r-1}S_n+{}_rC_3\cdot{}_{2r-3}S_n+{}_rG_5\cdot{}_{2r-5}S_n+\ldots).$$

58. Prove that

$$2.{}_5S_n+{}_3S_n=3({}_2S_n)^2 \text{ and } 7.{}_6S_n+5.{}_4S_n=12.{}_2S_n.{}_3S_n.$$

59. If w is an nth root of unity prove that $au+bw^{-1}$ satisfies

$$a^n+b^n=x^n-nabx^{n-2}+\frac{n(n-3)}{2!}a^2b^2x^{n-4}-\frac{n(n-4)(n-5)}{3!}a^3b^3x^{n-6}+\ldots$$

60. Prove that, when n is odd

$$x^n-nabx^{n-2}+\frac{n(n-3)}{2!}a^2b^2x^{n-4}-\frac{n(n-4)(n-5)}{3!}a^3b^3x^{n-6}+\ldots+a^n+b^n$$

$$=(x+a+b)\prod_{1}^{(n-1)/2}[x^2+2(a+b)x\cos(2r\pi/n)+a^2+b^2+ab\cos(4r\pi/n)]$$

and that when n is even

$$x^n-nabx^{n-2}+\frac{n(n-3)}{2!}a^2b^2x^{n-4}-\frac{n(n-4)(n-5)}{3!}a^3b^3x^{n-6}+\ldots-a^n-b^n$$

$$=[x^2-(a+b)^2]\prod_{1}^{(n-2)/2}[x^2+2(a+b)x\cos(2r\pi/n)+a^2+b^2+2ab\cos(4r\pi/n)].$$

---

61. Prove that the sum of those terms in the expansion of $(a+b+c)^r$ which contain every letter, r being a positiveinteger, is

$$(a+b+c)^r-[(b+c)^r+(c+a)^r+(a+b)^r]+a^r+b^r+c^r.$$

State the corresponding formula when there are n letters. Hence or otherwise prove that the number of ways of distributing r different gifts among n persons so that each receives at least one gift is

$$n^r-n(n-1)^r+\frac{n(n-1)}{2!}(n-2)^r-\frac{n(n-1)(n-2)}{3!}(n-3)^r+\ldots$$

[*B. U., 1924. Problems.*]

[Expand, in the notation of example 4, page 247 above, both sides of the identity

$$[F(a,x)-1][F(b,x)-1][F(c,x)-1] = F(a,x)F(b,x)F(c,x) - [F(b,x)F(c,x) + F(c,x)F(a,x) + F(a,x)F(b,x)] + F(a,x) + F(b,x) + F(c,x) - 1$$

in ascending powers of x and consider the coefficients of x^r on both sides of it.]

62. Prove that

$$\begin{vmatrix} x & a & 0 & 0 & b \\ b & x & a & 0 & 0 \\ 0 & b & x & a & 0 \\ 0 & 0 & b & x & a \\ a & 0 & 0 & b & x \end{vmatrix} = x^5 - 5abx^3 + 5a^2b^2x + a^5 + b^5.$$

63. Prove that $[1-(-x)^{n+1}]^n$ is equal to

$$\begin{vmatrix} c_0 & c_1x & c_2x^2 \ldots & c_{n-1}x^{n-1} & c_nx^n \\ c_nx^n & c_0 & c_1x \ldots & c_{n-2}x^{n-2} & c_{n-1}x^{n-1} \\ c_{n-1}x^{n-1} & c_nx^n & c_0 \ldots & c_{n-3}x^{n-3} & c_{n-2}x^{n-2} \\ \ldots\ldots & \ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ c_2x^2 & c_3x^3 & c_4x^4 \ldots & c_0 & c_1x \\ c_1x & c_2x^2 & c_3x^3 \ldots & c_nx^n & c_0 \end{vmatrix}$$

where $(1+x)^n = c_0 + c_1x + c_2x^2 + \ldots + c_nx^n$.

64. P, Q are points in the plane of a triangle ABC such that $\angle BAP = \angle QAC$ and $\angle CBP = \angle QBA$. Prove that $\angle ACP = \angle QCB$ and that $BC.AP.AQ + CA.BP.BQ + AB.CP.CQ = BC.CA.AB$. [Suppose A, B, C, P, Q, represent a, b, c, p, q in the Argand's diagram, then we have

$$\frac{p-a}{b-a} = l\frac{c-a}{q-a} \text{ or } \frac{(p-a)(q-a)}{(b-a)(c-a)} = l$$

where l is a real number. Similarly

$$\frac{(p-b)(q-b)}{(c-b)(a-b)}$$

is a real number equal to m say. Now observe that

$$\frac{(p-a)(q-a)}{(b-a)(c-a)} + \frac{(p-b)(q-b)}{(c-b)(a-b)} + \frac{(p-c)(q-c)}{(a-c)(b-c)} = 1 \ ]$$

65. Find the number and position of the real roots of $x^4-12x+5=0$. [*B U., 1927, Hons., I*]

66. Shew if II is the product of the squares of the differences of roots $a_1, a_2, \ldots a_n$ of the equation $f(x) \equiv x^n+px+1=0$.

$$(-1)^{\frac{1}{2}n(n-1)}\Pi = f'(a_1)\ f'(a_2)\ldots f'(a_n) = n^n q^{n-1}+(1-n)^{n-1}p^n.$$

[*R. U., M. A., 1926*]

$[\ f'(a_1) = na_1^{n-1}+p = a_1^{-1}(na_1^n+pa_1) = -a_1^{-1}[\ nq-p(1-n)a_1\ ]$

$\therefore\ f'(a_1)f'(a_2)\ldots f'(a_n) = q^{-1}[q^n n^n + pqnp^{n-1}(1-n)^{n-1}+qp^n(1-n)^n]$.

Further $f'(a_1) = (a_1-a_2)(a_1-a_3)\ldots(a_1-a_n)$ etc.]

67. Denoting $1^r-2^r+3^r-4^r+\ldots$ to n terms by ${}_rS_n'$ prove that

$$(-1)^{n-1}n^r(n+1)^r = 2({}_rC_{0.2r}S_n' + {}_rC_{2.2r-2}S_n' + {}_rC_{4.2r-4}S_n' + \ldots)$$

Deduce the following results:

$$2\ {}_2S_n' = (-1)^{n-1}n(n+1)\ .$$

$$2\ {}_4S_n' = (-1)^{n-1}n(n+1)[\ n(n+1)-1]\ .$$

$$2\ {}_6S_n' = (-1)^{n-1}n(n+1)[\ n^2(n+1)^2-3n(n+1)+3\ ]\ .$$

Prove also that ${}_{2r}S_n'$ is a polynomial in $n(n+1)$ divisible by $n(n+1)$

68. In the notation of the last example, prove that

$$4.{}_5S_n' + 8.{}_3S_n' = (-1)^{n-1}n^2(n+1)^2(2n+1)$$

69. Shew that $2\cos(3n+1)x-2(3n+1)\cos x-3n$ is divisible by $(2\cos x+1)^2$.

70. $A, B, C, \ldots$ are the vertices of a regular polygon of n sides inscribed in a circle with centre O and radius y. If P is a point in the plane of the circle such that $OP=x$ and $\angle POA=p$, prove that the sum of the angles which PA, PB, PC,.....make with OP, is

$$\tan^{-1}[y^n \sin np \div (y^n \cos np - x^n)]\ .$$

[Let O be the origin and OP, the axis of x in the Argand's diagram. Denote the vector OA by a; then the vectors OA, OB, OC,... are given by a, aw, aw^2....(where w is a special nth root of unity) and the vectors PA, PB, PC, ... are $a-x$, $aw-x$, aw^2-x, ... The product of these is equal to a^n-x^n or $y^n(\cos np+i\sin np)-x^n$. Now apply the second part of § 3·8. By applying the first part, we get De Moivre's Theorem (§ 7·71).]

71. If $(1+x)^n = c_0 + c_1x + c_2x^2 + \ldots + c_nx^n$

and $X = c_0 + c_3x^3 + c_6x^6 + c_9x^9 + \ldots$

$Y = c_1x + c_4x^4 + c_7x^7 + \ldots$

$Z = c_2x^2 + c_5x^5 + c_8x^8 + \ldots$.

prove that $(Y-Z)^2 + (Z-X)^2 + (X-Y)^2 = 2(1-x+x^2)^n$.

72. The first $n-r$ terms in the expansion of $(x+1)^n$ are given by

$$\begin{vmatrix} (x+1)^{n-1} & (x+1)^{r-1} & (x+1)^{r-1-1} & \ldots & (x+1)^2-1 & (x+1)-1 \\ {}_nC_r & {}_rC_r & 0 & \ldots & 0 & 0 \\ {}_nC_{r-1} & {}_rC_{r-1} & {}_{r-1}C_{r-1} & \ldots & 0 & 0 \\ {}_nC_{r-2} & {}_rC_{r-2} & {}_{r-1}C_{r-2} & \ldots & 0 & 0 \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots & \ldots & \ldots \\ {}_nC_1 & {}_rC_1 & {}_{r-1}C_1 & \ldots & {}_2C_1 & {}_1C_1 \end{vmatrix}$$

73. In the notation of example 71 above, prove that

$$c_0^{-1} - c_1^{-1} + c_2^{-1} - c_3^{-1} + \ldots + (-1)^r c_r^{-1} = \frac{n+1}{n+2}\left\{1 + \frac{(-1)^r}{{}_{n+1}C_{r+1}}\right\}.$$

and $1.c_0^{-1} - 2c_1^{-1} + 3c_2^{-1} - 4c_3^{-1} + \ldots + (-1)^r(r+1)c_r^{-1}$

$$= \frac{n+1}{n+3}\left\{1 + \frac{(-1)^r(r+2)}{{}_{n+1}C_{r+1}}\right\}.$$

Prove the following identities:

74. $(1+2a\cos x + a^2)(1+2a^2\cos 2x + a^4)(1+2a^4\cos 4x + a^8)\ldots$

$\times (1+2a^8\cos 8x + a^{16}) \times \ldots$ to n factors

$= (1-2a^l\cos lx + a^{2l}) \div (1-2a\cos x + a^2)$.

where $l = 2^n$.

75. $\dfrac{a\sin x}{1+2a\cos x+a^2} + \dfrac{2a^2\sin 2x}{1+2a^2\cos 2x+a^2} + \dfrac{4a^4\sin 4x}{1+2a^4\cos 4x+a^8}$

$+ \ldots$ to n terms $= \dfrac{a\sin x}{1-2a\cos x+a^2} - \dfrac{la^l\sin lx}{1-2a^l\cos lx+a^{2l}}$.

76. $\dfrac{a^2-1}{1+2a\cos x+a^2} + \dfrac{2(a^4-1)}{1+2a^2\cos 2x+a^4} + \dfrac{4(a^8-1)}{1+2a^4\cos 4x+a^8}$

$+ \ldots$ to n terms $= \dfrac{l(a^{2l}-1)}{1-2a^l\cos lx+a^{2l}} - \dfrac{a^2-1}{1-2a\cos x+a^2}$.

77. $(\cos x+\cos y)(\cos 2x+\cos 2y)(\cos 4x+\cos 4y)(\cos 8x+\cos 8y)$

$\ldots$ to n factors $= (\cos 2^n x - \cos 2^n y) \div (\cos x - \cos y)$.

78. $$\frac{(\sin x-\sin y)}{\cos x+\cos y}+\frac{2(\sin 2x-\sin 2y)}{\cos 2x+\cos 2y}+\frac{2^2(\sin 2^2x-\sin 2^2y)}{\cos 2^2x+\cos 2^2y}$$

$$+\ldots\text{to } n \text{ terms} = \frac{2^n(\sin 2^nx+\sin 2^ny)}{\cos 2^nx-\cos 2^ny} - \frac{\sin x+\sin y}{\cos x-\cos y}.$$

79. If $f(x)=0$ has k real roots and $F(x)=0$ has no real roots prove that $f(x)F'(x)-f'(x)F(x)=0$ has $k-1$ real roots at least and that $f(x)\,F''(x)-f''(x)F(x)=0$ has $k-2$ real roots at least. [*B. U., M. A., 1924*].

[If $P(x)\equiv f(x)F'(x)-f'(x)F(x)$ and $a_1, a_2, \ldots a_k$ are the real roots of $f(x)=0$ arranged in ascending order of magnitude, then $P(a_1)$, $P(a_2)$, ...have the same signs as $f'(a_1), f'(a_2), \ldots$ or $-f'(a_1), -f'(a_2)\ldots$ and hence they change alternately. Further observe that

$$P'(x)\equiv f(x)F''(x)-f''(x)F(x).]$$

80. Prove that $(m^2-n^2)\cos lx+(n^2-l^2)\cos mx+(l^2-m^2)\cos nx$ is divisible by $(\cos x-1)^2$, l, m, n being positive integers.

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81. Prove that $(1!2!3!\ldots n!)^2(-1)^{\frac{1}{2}n(n+1)}$ is equal to

$$\begin{vmatrix} n! & (n+1)! & (n+2)! & \ldots & (2n)! \\ (n-1)! & n! & (n+1)! & \ldots & (2n-1)! \\ (n-2)! & (n-1)! & n! & \ldots & (2n-2!) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1! & 2! & 3! & \ldots & (n+1)! \\ 1 & 1 & 1 & \ldots & 1 \end{vmatrix}.$$

82. Prove that ${}_nC_r$ is equal to

$$\begin{vmatrix} {}_nC_1 & {}_nC_2 & {}_nC_3 & \cdots & {}_nC_r \\ {}_{n-1}C_0 & {}_{n-1}C_1 & {}_{n-1}C_2 & \cdots & {}_{n-1}C_{r-1} \\ 0 & {}_{n-2}C_0 & {}_{n-2}C_1 & \cdots & {}_{n-2}C_{r-2} \\ 0 & 0 & {}_{n-3}C_0 & \cdots & {}_{n-3}C_{r-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & {}_{n-r+1}C_1 \end{vmatrix}.$$

Deduce a similar result true for all values of n.

83. a, b, c are three unequal quantities and a polynomial $f(x)$ is divided by $(x-a)(x-b)(x-c)$. Prove that the remainder $F(x)$ is given by

$$\begin{vmatrix} F(x) & x^2 & x & 1 \\ \Sigma f(a) & \Sigma a^2 & \Sigma a & 3 \\ \Sigma & \Sigma a^3 & \Sigma a^2 & \Sigma a \\ \Sigma a^2 f(a) & \Sigma a^4 & \Sigma a^3 & \Sigma a^2 \end{vmatrix} = 0.$$

84. The sides A_1A_2, A_2A_3, ... $A_{r-1}A_r$, A_rA_1 of a polygon $A_1A_2 \ldots A_{r-1}A_r$ make angles p_1, $p_2, \ldots p_{r-1}$, p_r with a given straight line and contain a_1, $a_2, \ldots a_r$, units of length. Prove that

$$a_1\cos p_1 + a_2\cos p_2 + \ldots + a_{r-1}\cos p_{r-1} + a_r\cos p_r = 0,$$
$$a_1\sin p_1 + a_2\sin p_2 + \ldots + a_{r-1}\sin p_{r-1} + a_r\sin p_r = 0,$$

and generally

$$\Sigma \frac{r!}{l!m!n!\ldots} a_1^{l} a_2^{m} a_3^{n} \ldots \cos(lp_1 + mp_2 + np_3 + \ldots) = 0$$

$$\Sigma \frac{r!}{l!m!n!\ldots} a_1^{l} a_2^{m} a_3^{n} \ldots \sin(lp_1 + mp_2 + mp_3 \ldots) = 0$$

l, m, n...being r positive integers (including zero) subject to the condition $l+m+n+\ldots = r$.

85. Prove that the equation

$$\frac{1}{n!} + \frac{x}{(n+1)!} + \frac{x^2}{(n+2)!} + \ldots + \frac{x^r}{(n+r)!} = 0$$

has one or no real root according as r is odd or even.

86. a_1, a_2, a_3, b_1, b_2, b_3 and l, m, n are real numbers and a_1, a_2, a_3, b_1, b_2, b_3 are arranged in order of magnitude. Prove that the roots of the following equation are all real:

$$\frac{l^2}{(x-a_1)(x-b_1)} + \frac{m^2}{(x-a_2)(x-b_2)} + \frac{n^2}{(x-a_3)(x-b_3)} = 0.$$

87. If $a_1x_1 + a_2x_2 + a_3x_3 + \ldots + a_{n-1}x_{n-1} + a_nx_n = l_1$
$a_2x_1 + a_3x_2 + a_4x_3 + \ldots + a_nx_{n-1} + a_1x_n = l_2$
$a_3x_1 + a_4x_2 + a_5x_3 + \ldots + a_1x_{n-1} + a_2x_n = l_3$
.....
$a_{n-1}x_1 + a_nx_2 + a_1x_3 + \ldots + a_{n-3}x_{n-1} + a_{n-2}x_n = l_{n-1}$
$a_nx_1 + a_1x_2 + a_2x_3 + \ldots + a_{n-2}x_{n-1} + a_{n-1}x_n = l_n$

and $f(x) = \dfrac{l_1 + l_2x + l_3x^2 + \ldots + l_nx^{n-1}}{a_1 + a_2x + a_3x^2 + \ldots + a_nx^{n-1}}$

prove that $nx_{s+1} = w^s f(w) + w^{2s} f(w^2) + w^{3s} f(w^3) + \ldots + w^{ns} f(w^n)$

where w is a special n th root of unity.

88. Prove that

$$\frac{n!(2\cos x)^r}{0!r!(n-r)!}+\frac{n!(2\cos x)^{r-2}}{1!(r-2)!(n-r+1)!}+\frac{n!(2\cos x)^{r-4}}{2!(r-4)!(n-r+2)!}+\dots$$
$$=2^n C_{r}.{}^n C_0\cos rx+2^n C_{r-1}.{}^n C_1\cos(r-2)x+2^n C_{r-2}.{}^n C_2\cos(r-4)x$$
$$+\dots.$$

89. Prove that

$$x^{2n}-i=(x^2-a^2)(x^2-a^{10})(x^2-a^{18})\dots(x^2-a^{8n-6})$$

where $a=\cos(\pi/4n)+i\sin(\pi/4n)$.

90. The n sides of a regular polygon are projected on a st. line. Prove that the product of the projections with their proper signs is

$$(-1)^n 2^{-n+2}d^n\sin\tfrac{1}{4}n(2x-\pi)\sin\tfrac{1}{4}n(2x+\pi)$$

where d is the side of the polygon and x, the angle which the given st. line makes with any one of the sides.

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* 91. Prove that the following polynomials are divisible by $F\equiv 1+x+x^2+\dots+x^{n-1}$, n being a prime number:

(*a*) $\Sigma_1, \Sigma_2, \dots, \Sigma_{n-1}$ and Σ_n+1

where Σ_r is the sum of products of

$$F-1,\ F-x,\ F-x^2,\dots,\ F-x^{n-1}$$

taken r at a time

(*b*) $$(F-1)^m+(F-x)^m+\dots+(F-x^{n-1})^m$$

where m is prime to n,

(*c*) $$(F-1)^m+(F-x)^m+\dots+(F-x^{n-1})^m-n(-1)^m$$

where m is a multiple of n.

(*d*) $$1^m+x^m+x^{2m}+\dots+x^{nm-2m}+(-1)^m(1+x+\dots+x^{n-2})^m$$

where m is prime to n.

In (*a*) Σ_{n-1} is also divisible by $F-x^{n-1}$.

In (*d*), if m is of the form $np+1$, the polynomial is divisible by F^2.

[Observe that all the above polynomials vanish for $x=w$ where w is a complex n th root of unity].

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* The Journel of the Indian Math. Society, question no 1407.

92. If polynomials $P(x)$, $Q(x)$, $R(x)$, $S(x)$ are each of the 3rd degree, prove that $3!\triangle=(a-x)^3\triangle'$ where $\triangle$, $\triangle'$ are

$$\begin{vmatrix} P(x), & Q(x), & R(x), & S(x), \\ P(a), & Q(a), & R(a), & S(a), \\ P'(a), & Q'(a), & R'(a), & S'(a), \\ P''(a), & Q''(a), & R''(a), & S''(a), \end{vmatrix}, \quad \begin{vmatrix} P(a), & Q(a), & R(a), & S(a), \\ P'(a), & Q'(a), & R'(a), & S'(a), \\ P''(a), & Q''(a), & R''(a), & S''(a) \\ P'''(a), & Q'''(a), & R'''(a), & S'''(a), \end{vmatrix}$$

93. Shew that the determinant of the $n+1$ th order

$$\begin{vmatrix} x^n & x^{n-1} & x^{n-2} & \ldots & x & 1 \\ 1 & 1 & 1 & \ldots & 1 & 1 \\ n_1 & (n-1)_1 & (n-2)_1 & \ldots & 1 & 0 \\ n_2 & (n-1)_2 & (n-2)_2 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & .. & \ldots & \ldots \\ n_{n-1} & (n-1)_{n-1} & 0 & \ldots & 0 & 0 \end{vmatrix}$$

is equal to $(x-1)^n(-1)^l$ where $l=\frac{1}{2}n(n-1)$.

94. O, A, B, C; are any four pts. in a plane; BC, CA, AB, OA, OB, OC are of lengths a, b, c and a', b', c' and make angles x, y, z, x', y' z' with a given direction. Prove that

$$aa'\cos(x+x')+bb'\cos(y+y')+cc'\cos(z+z')=0,$$

$$aa'\sin(x+x')+bb'\sin(y+y')+cc'\sin(z+z')=0.,$$

and that

$$a^2a'^2+b^2b'^2+c^2c'^2-2bb'cc'\cos l-2cc'aa'\cos m-2aa'bb\cos n=0$$

where l, m, n are equal to

$$<BAC-<BOC,\ <CBA-<COA,\ <ACB-<AOB.$$

Deduce Ptolemy's Theorem by taking the pt. O on the circum-circle of $\triangle ABC$.

95. Prove that the equation

$$1+\frac{n}{1!}x+\frac{n(n-1)}{2!}x^2+\ldots+\frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}x^r=0$$

has one or no real root according as r is odd or even, n being a real number greater than r or negative.

96. Prove that

$$\sum_{m=1}^{m=s}\sum_{n=1}^{n=r}\frac{m(m+1,(m+2)\ldots(m+n-1)}{n!}=\frac{(s+1)(s+2)\ldots(s+r+1)}{(r+1)!}$$

$$-s-1\ .$$

97. Prove that

$$\frac{1}{m}-\frac{n}{1}\cdot\frac{r}{m+r}+\frac{n(n-1)}{1.2}\frac{r^2}{m+2r}-\ldots+\frac{(-1)^n r^n}{m+nr}$$

$$=\frac{(1-r)^n}{m}+\frac{n(1-r)^{n-1}r^2}{m(m+r)}+\frac{n(n-1)(1-r)^{n-2}r^4}{m(m+r)(m+2r)}+\ \ldots\text{to } n+1 \text{ terms.}$$

[*B. U , 1918, Problems*].

[Prove by induction or integrate the identity

$$x^{m-1}(1-rx^r)^n\equiv x^{m-1}[\ (1-r)^n+{}_nC_1(1-r)^{n-1}r(1-x^r)$$
$$+{}_nC_2(1-r)^{n-2}r^2(1-x^r)^2+\ldots\ ],$$

between zero and one].

98. Prove that

$$\sec nx.\sec^n a.\cos(nx+a)=\prod_{0}^{n-1}\left\{\ 1-\tan x.\ \cot\frac{(2r-1)\pi-2a}{2n}\right\}.$$

99. A_0A_1, A_1A_2, A_2A_3, ... $A_{n-1}A_n$ are drawn parallel to the sides of a regular polygon and they contain 1, 2, 3, ... n units of length. Prove that $A_0A_n=\frac{1}{2}n\cdot\sec(\pi/n)$ and that A_0A_n makes with A_0A_1 an angle equal to $-(n+2)\pi/2n$.

100. A, B, C... and A_1, B_1, C_1, ... are the vertices of regular polygons of n sides inscribed in two circles with centres O and O_1 and radii a and a_1. If $OC_1=x$ and OA, O_1A_1 make angles p, p_1 with OO_1, prove that

$$AA_1^2.\ BB_1^2.\ CC_1^2\ldots=x^{2n}-2x^ny^n\cos nq+y^{2n}$$

and that the sum of the angles which AA_1, BB_1, CC_1, ... make with OO_1, is

$$\tan^{-1}\frac{y^n\sin nq}{y^n\cos nq-x^n}$$

where q and y satisfy the relations

$$a\sin(p-q)=a_1\sin(p_1-q)\ ;\ y^2=a^2-2aa_1\cos(p-p_1)+a_1^2.$$

101. ABC is a spherical triangle. Any point P on the sphere is joined to A, B, C and the cosines of these arcs are denoted by (a_1, b_1, c_1); similarly (a_2, b_2, c_2) (a_3, b_3, c_3), (a_4, b_4, c_4) denote the cosines of the arcs joining three other points Q, R, S to the vertices A, B, C. Prove that the four points P, Q, R, S lie on a small circle, if

$$\begin{vmatrix} a_1 & b_1 & c_1 & 1 \\ a_2 & b_2 & c_2 & 1 \\ a_3 & b_3 & c_3 & 1 \\ a_4 & b_4 & c_4 & 1 \end{vmatrix} = 0.$$

By applying the theorem to the polar triangle or otherwise, deduce a similar condition involving the sines of the distances of P, Q, R, S from the sides of $\triangle ABC$.

[Indian Math. Society's Journal, question no. 1319]

THEMES FOR NOTES AND ESSAYS.

1. Determinants and their applications to the solution of linear equations, elimination, and geometry.

2. Complex numbers and their applications to geometry and vector algebra.

3. De Moivre's Theorem and its immediate consequences.

4. Reciprocal equations.

5. Algebraic solution of equations with special reference to cubic and biquadratic equations.

6. Location and approximate evaluation of the real roots of numerical equations.

7. The properties of the n th roots of unity and their application to factorization. Mention their other uses.

8. Partial fractions and their uses.

9. Euler's Identity and the summation of finite series.

10. Van der Monde's Theorem and the summation of finite series.

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